ROBUSTNESS IN NETWORK FORMATION GAMES

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I propose a notion of equilibrium robustness, and characterize the robustness of Nash equilibria in a class of well-studied network formation games that suffers from multiplicity of equilibria. A Nash equilibrium is k-robust if k is the smallest integer such that the Nash equilibrium network can be perturbed by adding some k number of links. I show that this notion of robustness is analytically tractable – exact characterization and bounds for the robustness values are obtained. For instance, all acyclic Nash, with the exception of the periphery-sponsored star, are 1-robust, and are particularly fragile. Moreover among all Nash equilibria, cyclic or acyclic, the periphery-sponsored star has the highest robustness value. I also show and quantify that this most robust equilibrium is at least twice as robust as the next most robust equilibrium, asymptotically in large networks. The result holds true non-parametrically for the class of all convex benefit functions.

1. INTRODUCTION

In this paper, I propose a notion of equilibrium robustness, and analyze the robustness of Nash equilibria in a class of well-studied network formation games that suffers from multiplicity of equilibria. Robustness is defined as the minimum number of links that must be added to a Nash equilibrium network such that some agents will have incentives to deviate. I show that this notion of robustness is particularly tractable, and enables analytical characterization of robustness in many cases.

More precisely, a Nash equilibrium is k-robust if k is the smallest integer such that the Nash equilibrium network can be perturbed by adding some k number of links. In another words, for a k-robust Nash equilibrium, it is not possible to perturb the equilibrium network with fewer than k links, and there exists a

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way of perturbing the Nash equilibrium by adding some k links. The contribution of this paper is then to characterize the robustness values of equilibrium networks.

In the case of acyclic equilibrium networks, the robustness values are analytically and exactly characterized. While for general equilibrium networks, I derive upper bounds for the robustness values. An upper bound is derived for the case when equilibrium networks contain cycle of length at most 3, and when equilibrium networks contain cycle of length 4 or more.

These bounds are informative, and are subsequently used to obtain the following insights. **Firstly**, acyclic equilibrium networks are particularly fragile. With the exception of the periphery-sponsored star, all acyclic Nash equilibria are 1-robust, or minimally robust. Adding just one costless link is enough to perturb the equilibria. Thus this paper provides a new way to reconcile the fact that Nash equilibrium often predicts acyclic networks, and the fact that acyclic networks are not commonly observed in the real world.

Secondly, a particular kind of network, the periphery-sponsored star, is more robust than any other equilibrium network, cyclic or acyclic. Moreover I can quantify how much more robust is the periphery-sponsored star: asymptotically as the number of nodes grows, it must be *at least* **twice** as robust as any other equilibria, cyclic or acyclic. This finding complements Hojman and Szeidl (2008) and Feri (2007), where the periphery-sponsored star also emerges as a stable configuration. The novelty here is in quantifying that this most robust equilibrium is in fact, at least twice as robust as the next most robust equilibrium.

Now I will briefly describe the model of network formation, as well as the **intuition** behind the results. The setup here is the standard distance-based network formation game (Bala and Goyal (2000)). An agent in this game receives benefits from being connected to other agents in the network, but the benefit from being connected to another agent decays as the network distance between them increases. In addition, an agent can pay a fixed cost to unilaterally form a link, and thus faces a trade-off between minimizing costly link formation and maximizing connectedness. Multiplicity of Nash equilibria is a feature in this game.

Another important feature of this game is that a link in the network imposes positive externalities on other agents. Since links are costly, agents have incentives to free-ride on links that are formed by other agents. I will show that this proposed notion of robustness measures the extent to which there are *off-path incentives to free-ride*. For instance, although acyclic equilibria are stable under

Nash equilibrium, I claim that such equilibria must conceal rampant incentives to free-ride. Perturbing the equilibria with just one link is enough to increase the incentives of some agents to free-ride to the extent that they would then deviate from the Nash equilibrium strategy.

The setup here is actually *more general* in that no specific parametric form of the benefit function is assumed. As corollaries, I apply the results here to specific models of network formation games found in economics (Bala and Goyal (2000)) and computer science (Fabrikant et al. (2003)). Robustness values are derived for Nash equilibria of those models.

In Section 3.3, I discuss a *behavioral motivation* behind this notion of robustness. In particular, costless addition of links are motivated by the non-strategic and random aspect of network formation that is pervasive in the real-world, for instance, random meetings between friends-of-friends or between strangers (Jackson and Rogers (2007)).

1.1. Related Literature

Following the pioneering work of Jackson and Wolinsky (1996) and Bala and Goyal (2000), this paper builds on the literature of strategic network formation in economic theory. Comprehensive surveys on the subject can be found in Jackson (2008) and Goyal (2009).

In the economics literature, the two most closely related papers that deal with the problem of multiple equilibria are Hojman and Szeidl (2008) and Feri (2007). By placing plausible restrictions on the benefit function and the effect of decay, Hojman and Szeidl (2008) show that there is a unique Nash equilibrium, which is the periphery-sponsored star.[1] Using the notion of stochastically stable equilibrium (SSE), Feri (2007) also characterizes the periphery-sponsored star as the unique SSE.[2]

This paper differs in that the main result is on the robustness values of different Nash equilibria, obtaining exact values, and bounds in other cases. The main result quantifies and shows that the most robust equilibrium, the periphery-sponsored star, is at least twice as robust as the next most robust equilibrium network.

[1] Bloch and Dutta (2009) considers an interesting class of distance-based network formation games similar to Bala and Goyal (2000), with the innovation that agents can choose to form links with varying strengths. They also obtain the star networks as both the uniquely stable and efficient networks.

[2] Other papers that employ the notion of SSE are Goyal and Vega-Redondo (2005) and Jackson and Watts (2002a,b).

This notion of robustness delivers prediction that is consistent with the extant literature, but also enables further characterization and new insights.

The theoretical computer science literature also studies these strategic network formation games (Fabrikant et al. (2003) and Corbo and Parkes (2005)). To deal with equilibrium multiplicity, they define the Price of Anarchy as an indication of how bad Nash equilibrium performs in the worst case relative to the socially optimal outcome.[3]

1.2. EXAMPLE

Figure 1 below depicts a center-sponsored (CS) star network, where the center agent pays for all the links. This network is known to be stable under the Nash equilibrium (see Proposition 1). Another variant of star networks, the periphery-sponsored (PS) star, is depicted on the right-hand side of Figure 1. In the periphery-sponsored star, the periphery agents pay for links to the center instead.

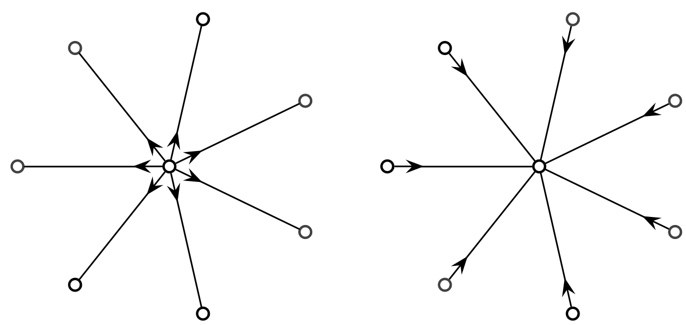

Figure 1: Center-sponsored star (Left) and Periphery-sponsored star (Right).

The notion of robustness propounded in this paper makes the following stark statement: while the periphery sponsored star is the most robust Nash equilibrium, all other star networks are minimally robust Nash equilibria (adding one link is enough to perturb the equilibrium).

To see this, observe that in a CS star, each one link that the center pays for connects him to just one agent. Now in a PS star, each periphery agent pays for a link that connects him to the center, *and* indirectly to all other agents. Therefore the periphery-to-center link provides a much higher marginal benefit to

[3]These papers provide bounds on the price of anarchy, a few papers followed suit in providing incrementally better bounds. See Chapter 19 of Vazirani et al. (2007).

the sponsoring node than the center-to-periphery link. Nash equilibrium does not discriminate between these two configurations, even though there is an intuitive sense in which the periphery-to-center link is more valuable to the sponsoring node, and hence should be the stronger and more robust type of link.

What heuristics allows us to say that links with higher marginal benefits should be formed? Consider adding a costless link to the center-sponsored star, as depicted in Figure 2. A costless link is formed between agents i and j where none of the agents have to pay for it.[4]

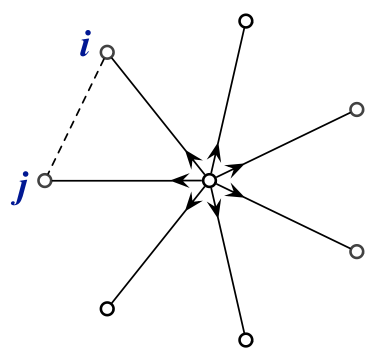

Figure 2: Pertubed network demonstrates off-path incentive to free-ride in the center-sponsored star.

Now the center agent can sever his costly link to the periphery agent i, and still connects to i through agent j. Therefore in this off-path network, the center agent's marginal benefit from having a link to the periphery agent i is now so low that the center agent prefers to dissolve this link to i – essentially free-riding on the costless link ij.

On the other hand, the periphery-to-center link provides such significant marginal benefit that even when perturbed with a new link, the periphery agent has no incentive to free-ride on this added link – if he severs his link to the center, he will be further away from the center agent, and also *all other periphery agents that he is not directly linked to*. In this sense, the periphery-to-center link is robust and there is no off-path incentive to free-ride.

[4]For instance, they meet and link with each other at random. Random network formation often coexists with strategic network formation as the driving forces behind network formation (Jackson and Rogers (2007)).

2. SETUP

The setup is a distance-based (unilateral) network formation game. A network formation game has agents (or nodes) $N = \{1, 2, \ldots n\}$. The action available to an agent i is as follows: agent i can choose any subset of agents, $s_i \in 2^{N \setminus \{i\}}$, and form a link to each agent $j \in s_i$ in that subset. The strategy space of agent i is then given by the set $S_i = 2^{N \setminus \{i\}}$. A strategy profile $s = s_1, \ldots, s_n \in S_1 \times \cdots \times S_n$ induces a directed network (N, g) where $g = \cup_{i=1}^{n} \{(i, j) : j \in s_i\}$ is a collection of ordered pairs representing the edges that are present in the network. When it is clear, we will refer to a network (N, g) as just g.

Given a directed network g, let us define the corresponding **undirected** network of g by $\hat{g}$, where there is an undirected link $\{i, j\} \in \hat{g}$ if either $(i, j) \in g$ or $(j, i) \in g$. Denote $d(i, j; \hat{g})$ as the distance between nodes i and j in the undirected network $\hat{g}$, i.e. the length of the shortest path from i to j in the undirected network $\hat{g}$.

Finally, let $b : \{1, \ldots, n-1\} \to \mathbb{R}$ be the **benefit function**. The function b models the benefit that an agent receives from another agent as a function of the distance between them. The benefit that agent i receives from agent j in the network g is $b[d(i, j; \hat{g})]$. While link formation is one-sided, the notion of distance and benefit between two nodes are always undirected.

When the strategy profile is s, the payoff of agent i is given by:

$$u_i(s) = \sum_{j \in N \setminus \{i\}} b[d(i, j; \hat{g}_s)] - c|s_i| \tag{1}$$

where $\hat{g}_s$ is the undirected network induced by the strategy profile s. This payoff function is exactly the one assumed in Section 11.3.2 of Jackson (2008).

The first term of Equation (1) is the sum of all benefits that agent i receives in the network g_s, while the second term is the total cost paid for by agent i in forming $|s_i|$ number of links, and where each link costs $c > 0$.

We will assume that an agent receives zero benefit from agents he has no connection to, i.e. $b(\infty) = 0$. Moreover, we will assume monotonicity of the benefit function, i.e. $b(k) \geq b(k+1)$ for any k. A common benefit function found in the literature is $b(x) = \delta^x$ where the discount or decay factor δ is in the set $(0, 1]$ (Bala and Goyal (2000)). Agents obtain utility from their direct connections and also from their indirect connections, but the benefit decays with the distance between individuals. Overall, the utility function in Equation 1 captures the competing incentives of the player to maximizing connections to other agents, while minimizing forming costly links.

A directed network g encodes all the payoff-relevant information. We can write $u_i(g)$ to denote the utility that agent i derives from the network g, where $u_i(g) = \sum_{j \neq i} b[d(i,j;\hat{g})] - c \cdot d_{out}(g)$. Here, $d_{out}(g)$ is the out-degree of agent i in g, and $\hat{g}$ the undirected network of g.

2.1. Nash equilibrium

A (pure) **Nash equilibrium** of the game is an $s = (s_i, s_{-i})$ such that, for each player i, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$. A Nash strategy profile s induces a network g, and for convenience, we will often say that g is a Nash equilibrium network, although it should be clear that Nash equilibrium refers to the underlying strategy profile. In a Nash equilibrium, each player maximizes his or her own utility, without taking into account the positive externality that own's action has on other agents.

In contrast, a socially optimal or **efficient** network is one that maximizes the overall welfare and total payoffs in the network, giving equal weight to each of the player's utility. Formally, A strategy profile s and its induced network g is efficient relative to a profile of utility functions $(u_1, \ldots, u_n)$ if $\sum_i^n u_i(s) \geq \sum_i^n u_i(s')$ for all $s' \in S$.

Definition 2.1. A **periphery-sponsored star** network on n agents is a configuration where $n-1$ agents (the periphery nodes) each pays for a costly link to one node (the center node). These $n-1$ links are the only links in the network. Similarly, in a **center-sponsored star** network, the center node pays for costly links to each of the $n-1$ agents.

Proposition 1 below characterizes the Nash equilibrium of the game. The proof is postponed to the Appendix. Note that the same proposition appeared as Proposition 11.4 in Jackson (2008), but with a typo that was later corrected.[5] Proposition 1 in this paper is the corrected version.

Proposition 1 (Nash Equilibrium)**.**

(i) For $b(1) - b(2) > c$, the unique Nash equilibrium of the game is the complete network.

[5] http://web.stanford.edu/~jacksonm/netbook-updates.pdf

(ii) *For* $b(1) - b(2) < c < b(1)$*, any star is a Nash equilibrium but there are other equilibria.*

(iii) *For* $b(1) < c < b(1) + b(2)(n-2)$*, the empty network is a Nash equilibrium. The periphery-sponsored star is the only type of star that can be sustained in a Nash equilibrium. There are also other possible non-star Nash equilibria.*

(iv) *For* $b(1) + b(2)(n-2) < c$*, the empty network is the unique Nash equilibrium.*

Lemma 2 below is an additional characterization of Nash equilibria, which will later be used to prove the main theorems. It says that in any Nash equilibrium network, either there is a no link (empty network), or all agents are connected in the sense that there is a path between any two agents in the network. In another words, either there is exactly n components, or there is exactly one component in any Nash network. To the best of my knowledge, this has not been mentioned elsewhere. The proof is available in the Appendix.

Lemma 2. *If a Nash equilibrium is nonempty, then it must be connected*[6]*, i.e. there is exactly one component in the undirected network induced by the Nash equilibrium.*

For completeness, the characterization of efficient networks are restated below, which can be found as Proposition 11.3 in Jackson (2008).

Proposition 3 (Efficient Network)**.**

(i) *For* $2(b(1) - b(2)) > c$*, the unique efficient outcome is the complete network.*

(ii) *For* $2(b(1) - b(2)) < c < 2b(1) + b(2)(n-2)$*, the star network is the unique efficient outcome*

(iii) *For* $2b(1) + b(2)(n-2) < c$*, the empty network is the unique efficient outcome.*

3. Notion of Robustness

In this section, I will discuss the issue of multiple equilibria and introduce a new notion of equilibrium robustness.

[6]An undirected network g is connected if there is a path between any two agents in the network, i.e. for any pair of agents i, j, there is a sequence of agents $a_1, \dots, a_m$ such that $\{i, a_1\}$ is a link in g, $\{a_k, a_{k+1}\}$ is a link in g for $k = 1, \dots, m-1$, and $\{a_m, j\}$ is a link in g.

3.1. Multiplicity of Nash equilibrium

In Proposition 1, we saw that when the cost of maintaining a link lies in an intermediate region, multiple networks arise as Nash equilibria.

Recall that the complete network and the empty network are the unique Nash equilibrium for $c < b(1) - b(2)$ and $c > b(1) + (n-2)b(2)$ respectively. Since there is no issue with equilibrium selection in this region, we will henceforth only be concerned with the intermediate range of c, i.e. $b(1) - b(2) < c < b(1) + (n-2)b(2)$, where the issue of multiplicity arises.

To starkly illustrate the problem of multiple equilibria, consider the region of cost high enough that the empty network is a Nash equilibrium. Specifically consider the region $H = [c : b(1) < c < b(1) + b(2)(n-2)]$. It turns out that even in the region $c \in H$, both the empty network as well as super-connected networks can arise as Nash equilibria. A **super-connected** network is one where the network remains connected upon the deletion of any one link. For example, the super-connected cycle of length 5 in Figure 3 is a Nash equilibrium for $b(1) - b(2) < c < b(1) - b(4) + b(2) - b(3)$. This interval overlaps with with the region H when the benefit function b declines rapidly enough such that $b(2) - b(3) - b(4) > 0$.

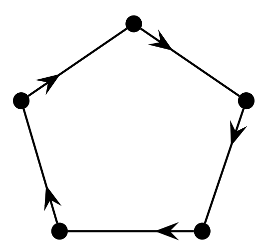

Figure 3: An equilibrium that is super-connected

3.2. Robustness

Our notion of robustness is based on perturbing an equilibrium network with **costless links**. More formally, define $g \cup \{i, j\}$ as adding an undirected link $\{i, j\}$ to the directed network g. The undirected link $\{i, j\}$ models a costless link, where no agent is paying for the link, but all agents can benefit from the link.

I now define the robustness value of an equilibrium network as follows: *what is the minimum number of costless links required to perturb the Nash equilibrium? More concretely, a Nash equilibrium is K-robust if (i) there exists a way of adding K or more costless links such that some agents now have incentives to deviate from their Nash equilibrium strategy, and (ii) for all $k < K$, there does* **not** *exist a way of adding k costless links such that some agents prefer to deviate from their Nash strategy.*

The formal definition of K-robust Nash equilibrium is given below, but first we need a notation. A network g' is **obtainable** from a network g by player i if the only changes between networks g' and g involve links that are directed from i to other agents. Formally, a network g' is obtainable from a network g by player i if $[(k,j) \notin g'$ and $(k,j) \in g]$ or $[(k,j) \in g'$ and $(k,j) \notin g]$ implies that $k = i$.

Definition 3.1 (K-robustness)**.** A Nash equilibrium s^* and its corresponding network g is a K**-robust** Nash equilibrium if and only if

(i) There exists K pairs of nodes $\{\{i_t, j_t\}_{t=1}^{K}\}$, such that when $\{\{i_t, j_t\}_{t=1}^{K}\}$ are added as costless links to (N, g), there exists a player i who has the incentive to deviate from the Nash equilibrium strategy s^*. Formally, there exists $i \in N$ such that,

$$u_i(g^i \cup \{i_t, j_t\}_{t=1}^{K}) > u_i(g \cup \{i_t, j_t\}_{t=1}^{K})$$

where g^i is obtainable from g by player i.

(ii) For all $k \in \{1, 2, \ldots, K-1\}$, there does not exist $\{\{i_t, j_t\}_{t=1}^{k}\}$ such that

$$u_i(g^i \cup \{i_t, j_t\}_{t=1}^{k}) > u_i(g \cup \{i_t, j_t\}_{t=1}^{k})$$

for some $i \in N$, where g^i is obtainable from g by player i.

In words, a Nash equilibrium is K-robust if K is the **minimum** number of costless links needed to perturb the Nash equilibrium. We will not be able to perturb the given Nash equilibrium with fewer than K costless links. If an equilibrium network is 1-robust, I will say that it is **minimally robust**, where the addition of one costless link between some two nodes will result in some agents deviating from their Nash strategy. When no such K exists, the equilibrium is maximally robust.

3.3. Behavioral motivation

I will briefly remark on how k-robustness can be motivated from a behavioral perspective. It has been recognized that real-world networks are partly driven by a random, non-strategic aspect of link formation (Jackson and Rogers (2007)). Links are routinely formed between individuals without their costs and benefits taken into consideration.

These random meetings between friends-of-friends or between stangers motivate costless addition of links to the network. On the other hand, we expect strategic agents to react to these random meetings. Hence, our notion of robustness intuitively refines away networks that are not stable under this interplay between strategic and non-strategic forces.

4. Robustness of acyclic networks

The proposed notion of robustness is tractable in that exact robustness values can be derived for the class of all acyclic Nash equilibria. All acyclic Nash equilibria, except the periphery-sponsored star, is 1-robust. While the periphery-sponsored star itself has a robustness value that can be calculated according to Proposition 4 below.

4.1. Robustness of periphery-sponsored stars

Proposition 4. *The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where*

$$m = \max\left\{\frac{b(1) - b(2) + (b(2) - b(3))(n-2) - c}{b(2) - b(3)}, 1\right\}$$

Proof: In a periphery-sponsored star with n agents, the $n-1$ periphery agents each pays c for a link to the center agent. Now consider perturbing this equilibrium network by adding a costless link between some two agents. This costless link must be between two periphery agents, since the center agent is already linked to everyone else. No agent will respond to this perturbation by forming more links (otherwise the complete network is an equilibrium), nor will agents respond by switching link from the center agent who is linked to the maximum number of agents to another agent. The only possible response of a periphery agent is that he might sever his existing link to the center.

A periphery agent i would have incentive to sever his link to the center when enough costless links are formed between him and the other periphery agents.

Therefore consider adding t links between agent i and $\{1,\ldots,t\}$ other periphery nodes. Agent i's marginal benefit from his link to the center is now $b(1)-b(2)+(n-2-t)(b(2)-b(3))$. To see this, without the link to the center, agent i would experience an increase in distance of 1 unit to each of the $(n-2-t)$ periphery node that he does not have a direct link to, which corresponds to a decrease in utility of $(n-2-t)(b(2)-b(3))$; as well as an increase in distance of 1 to the center (decrease in utility of $b(1)-b(2)$).

So the perturbed network is no longer a Nash equilibrium if the marginal benefit of agent i's link to the center is now less than c, that is, $b(1)-b(2)+(n-2-t)(b(2)-b(3)) \leq c$, or after rearranging, $t \geq \frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}$. Now observe that this is the most efficient way to perturb the periphery-sponsored star, in the sense that adding links any other ways would require more links for perturbation. To see this, if we add t number of costless links in such a way that only $\tilde{t} < t$ of them are linked to i, then agent i's marginal benefit from his link to the center is $b(1)-b(2)+(n-2-\tilde{t})(b(2)-b(3))$, which is strictly less than $b(1)-b(2)+(n-2-t)(b(2)-b(3))$. Since agent i is at most 2 links apart from every other nodes, any added costless links that do not involve i will not affect i's marginal benefit of linking to the center.[7]

Therefore, $\max\left\{\frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}, 1\right\}$ is the minimum number of links needed to perturb the periphery-sponsored star.

□

4.2. Application

To grasp the magnitude of the robustness value in Proposition 4, let us consider a special case of the distance-based utility model in which the benefit function is linear in distance. That is, consider the following objective function obtained from substituting $b(x) = -x$ into Equation (1): $u_i(s) = \sum_{j\neq i}^{n} -d(i,j;\hat{g}) - c|s_i|$

where $d(i,j;\hat{g})$ is the shortest path length between i and j in the undirected network $\hat{g}$ induced by the strategy profile s. This class of unilateral network formation games was first studied in Fabrikant et al. (2003), where they developed and quantified the Price of Anarchy for this class of games.

[7] In fact, when t costless links are added between agent i and $\{1,\ldots,t\}$ other periphery agents, the marginal benefit of the link from $j \in \{1,\ldots,t\}$ to the center is also $b(1)-b(2)+(n-2-t)(b(2)-b(3))$. So if agent i prefers to deviate, these $\{1,\ldots,t\}$ agents also prefer to deviate.

Corollary 5: *Consider the network formation game of Fabrikant et al. (2003), where the benefit function is $b(x) = -x$. The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where $m = \max\{n - c - 1, 1\}$*

As another example, we will consider the formulation of network formation game in Bala and Goyal (2000), where the benefit function is set to the convex function: $b(x) = \delta^x$ for some $\delta \in (0, 1]$.

Corollary 6: *Consider the network formation game of Bala and Goyal (2000), where the payoff function given by $u_i(s) = \sum_{j \neq i}^{n} \delta^{d(i,j;\hat{g})} - c|s_i|$. The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where $m = \max\{n + f(c, \delta), 1\}$ and $f(c, \delta) = \frac{1}{\delta} - 1 - \frac{c}{\delta^2(1-\delta)}$.*

4.3. Robustness of acyclic equilibria

The following lemma shows rather strikingly that with the exception of the periphery-sponsored star, all acyclic equilibria are minimally robust. The addition of just one link is sufficient to perturb an acyclic Nash equilibrium.

An acyclic equilibrium is one where the Nash strategy profile induces an **undirected acyclic** network. An undirected network is **acyclic** if and only if there does not exist a sequence of nodes $(a_1, \ldots, a_k)$ such that there is a link between nodes a_i and a_{i+1} for all $i = 1, \ldots, k-1$, and there is a link between nodes a_k and a_1.

For notational convenience, $g - ij$ is the network obtained from g by removing the directed link (i, j) between agents i, j; and $g + ij$ is the network obtained from g by adding the directed link (i, j).

Theorem 7. *All* **acyclic** *Nash equilibrium networks are 1-robust, or minimally robust, with the exception of the periphery-sponsored star.*

Proof: Take any path of length 2 in an acyclic equilibrium network g, as depicted in Figure 4. Denote the agents as i, j, k, and suppose that agent j pays for the link to i. The key observation is the following equation 2:

$$(2) \qquad u_k(g + ki) - u_k(g) = u_j(g + ki) - u_j(g + ki - ji)$$

Equation 2 says that the marginal benefit of the link (k, i) to k in the network g is the same as the marginal benefit of the link (j, i) to j in the perturbed network $g + ki$ where the link (k, i) is added to g.

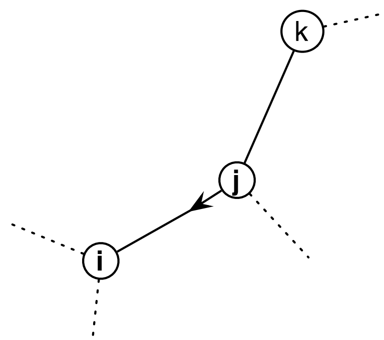

Figure 4: Acyclic equilibrium network, g

To show Equation 2, let C be the component containing node i that obtains when the link (j, i) is removed from g. Let $p(x)$ denote the shortest path from node i to node x in the component C. Now $p(x)$ is uniquely defined because C is acyclic. Observe that in the network g, the shortest path between k and $x \in C$ takes the form $(k, j, i, p(x))$, and when the link (k, i) is added to g, this path shortens to $(k, i, p(x))$. Similarly in the network $g + ki - ji$, the shortest path between j and $x \in C$ takes the form $(j, k, i, p(x))$, which shortens to $(j, i, p(x))$ in the network $g + ki$ when the link (j, i) is added to $g + ki - ji$.

Following Equation 2, since g is a Nash equilibrium, we have $u_k(g + ki) - u_k(g) < 0$, which implies $u_j(g + ki) - u_j(g + ki - ji) < 0$. Therefore when g is perturbed by adding just *one* costless link, between agents i and k, the best response of agent j is to sever the link to i. As a result, all acyclic equilibrium network with the structure depicted in Figure 4 must be 1-robust, or minimally robust.

For an equilibrium network to be more than minimally robust, any path of length 2 must have the configuration depicted in Figure 5. That is, take any path ijk, all the other agents must pay for links to the middle node j.

We know from Lemma 2 that an equilibrium network is either connected or empty. The only connected, acyclic network with this property is the periphery-sponsored star. Hence, all acyclic equilibria with the exception of the periphery-sponsored star are minimally robust. [8] □

[8] From Lemma 4, the periphery-sponsored star itself is minimally robust when the cost of link formation is high enough, i.e. when $c > b(1) - b(2) + (b(2) - b(3))(n - 3)$, in which case, all acyclic equilibria are minimally robust.

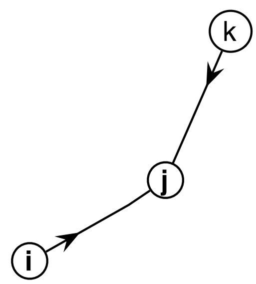

Figure 5: Acyclic equilibrium

5. Robustness of general networks

In this section, I derive upper bounds for the robustness values of Nash networks with cycles of length 3 and above. These bounds are informative – we can use these bounds to say that the periphery-sponsored star is the most robust equilibrium, and it is *at least* **twice** as robust as the next most robust equilibrium.

A standard assumption imposed on the benefit function is convexity. For this section, we will need the benefit function to be convex.

Assumption 1. *The benefit function is* **convex**, *i.e. the function b satisfies $b(k) - b(k-1) \geq b(k') - b(k'-1)$ for $k' \geq k$.*

Convexity or equivalently, diminishing marginal benefits is a natural assumption, as it says that agents care less about shortening his distance by 1 unit to an agent who is already far apart. The marginal benefit from reducing distance to another agent decreases as the distance to that agent increases. Convexity of the benefit function is satisfied by existing models in the literature. In particular, the benefit functions $b(x) = \delta^x$ of Bala and Goyal (2000) and $b(x) = -x$ of Fabrikant et al. (2003) are convex.

A Nash equilibrium has a cycle of length K whenever the Nash strategy profile induces an undirected network g such that there exists a sequence of agents $(a_1, \ldots, a_K)$ where there is a link between agents a_k, a_{k+1} in g for $k = 1, \ldots, K-1$ and there a link between agents a_1 and a_K in g.

Theorem 8. *Any Nash equilibrium network that has a cycle of length 3 is at most*

$\lceil \theta \rceil$*-robust, where*

$$\theta = \max\left\{ \frac{b(1) - b(2) + (b(2) - b(3))\lfloor \frac{n-3}{2} \rfloor - c}{b(2) - b(3)}, 1 \right\}$$

Corollary 9: *Consider the benefit function $b(x) = -x$, as in Fabrikant et al. (2003). Any Nash equilibrium that has a cycle of length 3 is at most $\lceil \theta \rceil$-robust, where $\theta = \max\left\{ \left\lfloor \frac{n-1}{2} \right\rfloor - c, 1 \right\}$*

THEOREM 10. *Any Nash equilibrium network that has a cycle of length 4 or above is at most $\lceil \theta \rceil$-robust, where*

$$\theta = \max\left\{ \frac{b(1) - b(2) + (b(2) - b(3))\lfloor n/2 \rfloor - c}{b(2) - b(3)}, 1 \right\}$$

Corollary 11: *Consider the benefit function $b(x) = -x$, as in Fabrikant et al. (2003). Any Nash equilibrium that has a cycle of length 4 or above is at most $\lceil \theta \rceil$-robust, where $\theta = 1 + \max\left\{ \lfloor \frac{n}{2} \rfloor - c, 0 \right\}$.*

Using these bounds, I show that the periphery-sponsored star is the unique most robust Nash equilibrium. The periphery-sponsored star (along with other star networks) is also the most efficient network (Proposition 3).

THEOREM 12. *For costs within the non-trivial range of $b(1) - b(2) < c < b(1) + b(2)(n-2)$, the periphery-sponsored star network is the unique most robust Nash equilibria. For $c > b(1) + b(2)(n-2)$, the empty network is the only Nash equilibrium. For $c < b(1) - b(2)$, the complete network is the unique Nash equilibrium.*

This main theorem follows from Proposition 4, as well as Theorems 7, 8 and 10 above. The proofs of Theorems 8 and 10 are provided in Section 6. More precisely, from Theorems 8 and 10, we know that all Nash equilibria containing some cycles have robustness value bounded above by $\lceil \tau \rceil$, with $\tau = \max\left\{ \frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2 \rfloor - c}{b(2)-b(3)}, 1 \right\}$. On the other hand, Propositions 4 and 7 state that all acyclic equilibria are minimally robust, with the exception of the periphery-sponsored star, which has a robustness value of $\lceil \theta \rceil$, where $\theta = \max\left\{ \frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}, 1 \right\}$. Since this value is greater than the upper bound of the robustness values of all cyclic Nash networks, it follows that the periphery-sponsored star is the most robust equilibrium.

5.1. MAGNITUDES

Interestingly, other papers have also claimed that the periphery-sponsored star stands out from other equilibrium networks as being a uniquely stable configuration (Hojman and Szeidl (2008) and Feri (2007)). The novelty here is to *quantify* and show that this is true in a very strong sense. This most robust equilibrium must be at least **twice as robust** as the next most robust equilibrium, asymptotically as the number of agents n grows (other parameters stay constant). Note that this statement is true for all convex benefit functions.

Proposition 13. *In the limit as $n \to \infty$, the periphery-sponsored star is at least* **two** *times more robust than any other Nash equilibrium. This magnitude does not depend on the choice of benefit function, and it is true for any convex benefit function.*

PROOF: Recall that the periphery-sponsored network is $\lceil m \rceil$-robust, where $m = \max\left\{\frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}, 1\right\} = n + O(1)$. From Theorems 8 and 10, the robustness of Nash networks containing cycles are bounded above by $\lceil \theta \rceil$, where $\theta = \max\{\frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2 \rfloor - c}{b(2)-b(3)}, 1\} = \lfloor \frac{n}{2} \rfloor + O(1)$. Therefore, the periphery-sponsored star is at least two times more robust than any other Nash equilibrium. □

6. MAIN PROOFS

Definition 6.1. For the proof, we would need the following definitions.

(i) A path in an undirected network g between nodes i and j is a sequence of links or links $i_1 i_2 \dots i_{K-1} i_K$ such that $(i_k, i_{k+1}) \in g$, for all $k = 1, \dots K-1$, with $i_1 = i$ and $i_K = j$. The length of the path is K.

(ii) $L(i,j;g) = \{l_1, \dots, l_L\}$ is the set of nodes for which **all** the shortest paths from i to $l \in L(i,j;g)$ contains the node j in the **undirected** network of g.

The set L has the following meaning: when nodes i and j are linked, then the set $L(i,j;g)$ is exactly (and no more) those nodes in which the distance from i to each node in $L(i,j;g)$ would increase when the link ij is removed from g.

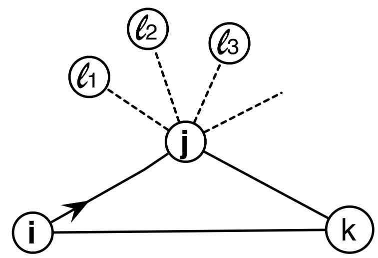

Figure 6: Equilibrium with a cycle of length 3

6.1. PROOF OF THEOREM 8

PROOF: Denote the Nash equilibrium network by g. Consider a cycle of length 3 in g involving the nodes $\{i, j, k\}$, as shown in Figure 6. Without loss of generality, assume that the link ij is paid for by i. Now the set $L(i, j; g)$ must be nonempty, otherwise the marginal benefit of the link ij to i is $b(1) - b(2)$, which is less than c by assumption, and i would strictly prefer not to form the link to j. Let $L(i, j; g) = \{l_1, l_2, \ldots l_{L_1}\}$, and $L_1 = |L(i, j; g)|$.

Consider adding t links between i and some t nodes from the set $L(i, j; g) = \{l_1, l_2, \ldots l_{L_1}\}$. By convexity of b, we have $b(k) - b(k-1) \geq b(k') - b(k'-1)$ for $k' \geq k$, it then follows that player i's marginal benefit of the link ij in this perturbed network is bounded above by $b(1) + (b(2) - b(3))(L_1 - t)$. When $c > b(1) + (b(2) - b(3))(L_1 - t)$, player i's best-response is to delete the link to j. Solving for t, we obtain $t > \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)}$. Therefore, with $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil$ costless links, we can perturb the equilibrium network g.

Now the proof technique is then to express the possible range of L_1 in terms of parameters of the model, n and c. A very crude upper bound for L_1 is just $n - 3 \geq L_1$, because there are n nodes and $L(i, j; g) \cap \{i, j, k\} = \emptyset$. We can do better:

Now without loss of generality, assume that the link between i and k is paid by k in equilibrium, as in Figure 7. The proof remains the same if we assume that i pays for the link ik. Now consider the set $L(k, i; g)$. If $|L(k, i; g)| = 0$, then k would not form link to i in equilibrium, so let $L(k, i; g) = \{m_1, \ldots, m_{L_2}\}$ with $|L(k, i; g)| = L_2 > 0$. Using the same reasoning as before, if $\lceil \frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)} \rceil$ costless links are formed between k and some $\{m_1, \ldots, m_{L_2}\}$, then by deleting the link ki, player k could strictly increase his payoff.

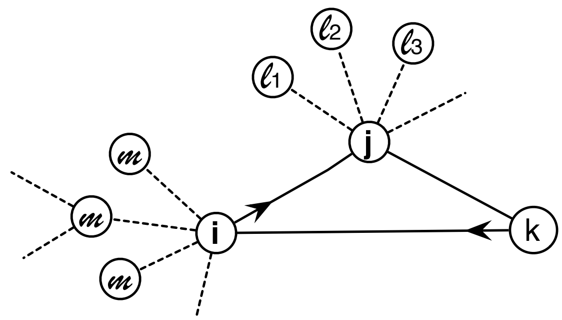

Figure 7: Cyclic equilibrium

Moreover, it must be the case that $L(i,j;g)$ and $L(k,i;g)$ are disjoint. Otherwise, if $x \in L(i,j;g) \cap L(k,i;g)$, then $ij\ldots x$ and $ki\ldots x$ are both shortest paths in the network g respectively. This implies that $kij\ldots x$ is the shortest path between k and x, but clearly $kj\ldots x$ is a shorter path, which gives a contradiction. Hence, we must have $n-3 \geq L_1 + L_2$

We now know that it does not take more than $\min\{\lceil\frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)}\rceil, \lceil\frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)}\rceil\}$ costless links to disrupt the equilibrium, where $n-3 \geq L_1+L_2$, and $L_1, L_2 > 0$. Therefore this number is an upper bound for the robustness value. To get rid of L_1 and L_2, we then find L_1, L_2 that maximize $\min\{\lceil\frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)}\rceil$, $\lceil\frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)}\rceil\}$ subject to $n-3 \geq L_1+L_2$. The maximum occurs at $L_1 = L_2 = \lfloor\frac{n-3}{2}\rfloor$. Therefore, any Nash equilibrium that has a cycle of length 3 is less than $\lceil\theta\rceil$-robust, where

$$\theta = \max\left\{\frac{b(1)-b(2)+(b(2)-b(3))\lfloor\frac{n-3}{2}\rfloor - c}{b(2)-b(3)}, 1\right\}$$

□

6.2. PROOF OF THEOREM 10

PROOF: Consider a network g induces by some Nash equilibrium. There are two distinct possibilities, either g has an (undirected) cycle of length 3, in which case we can simply invoke Theorem 8, or the network is triangle-free and has no cycle of length 3. We will then only consider a triangle-free network.

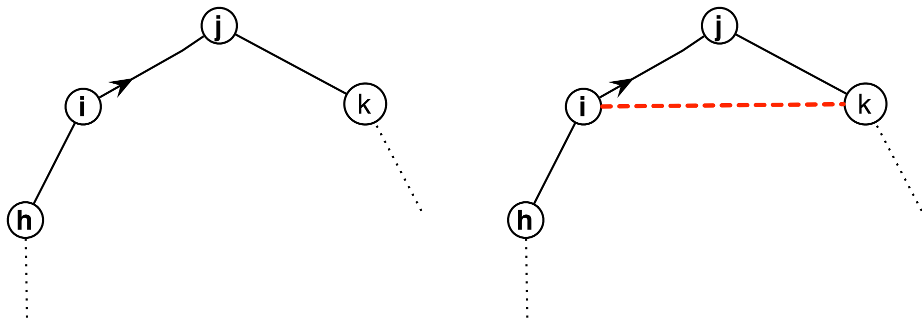

Figure 8: Perturbing an equilibrium with an (undirected) cycle of length greater than 3. Adding a costless link between i and k creates a cycle of length 3, and from there we can use Theorem 8.

Now let us consider a cycle of length 4 or above in this triangle-free network. This cycle is depicted in Figure 8. Without loss of generality, let node i pays for the link ij in equilibrium. Now if we add a costless link between i and k, then a cycle of length 3 emerges, and the problem is essentially transformed into the one encountered previously in Theorem 8. From Theorem 8, it suffices to add $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil + 1$ such that the link ij is perturbed, where $L_1 = |L(i,j;g+ik)|$. There is an extra term of $+1$ because we need to also count the costless link ik that creates a cycle of length 3 in the first place. In another words, we now have $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil + 1 = \lceil \frac{b(1)+(b(2)-b(3))(L_1+1)-c}{b(2)-b(3)} \rceil$ as an upper bound.

To improve on the upper bound, consider the link between h and i. Consider the *first* case where i is paying for the link to h, as shown in Figure 9. Adding a costless link between h and j as depicted in Figure 9 would create a cycle of length 3, and by invoking Theorem 8, we know that $\lceil \frac{b(1)+(b(2)-b(3))(L_2+1)-c}{b(2)-b(3)} \rceil$ is another upper bound, where $L_2 = |L(i,h;g+hj)|$.[9] Since the sets $L(i,h;g+hj)$ and $L(i,j;g+ik)$ are disjoint, the worst case upper bound occurs when $L_1 = L_2 = \lfloor \frac{n-4}{2} \rfloor$.

[9]Here, $g+hj$ denotes the network obtained from g by adding the costless link hj.

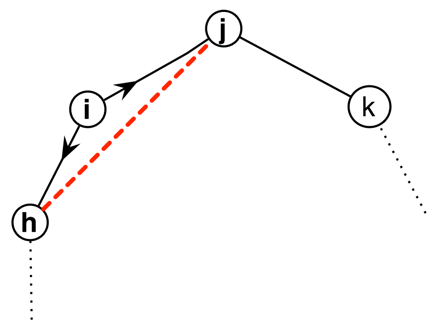

Figure 9: Case I – node i pays for the link to node h.

Finally, consider the second case where the node h is paying for the link to i, as depicted in Figure 10. Similar to the paragraph before, adding a costless link between h and j as depicted in Figure 10 would produce a cycle of length 3, and we can immediately deduce that $\lceil \frac{b(1)+(b(2)-b(3))(L_2'+1)-c}{b(2)-b(3)} \rceil$ is an upper bound, where $L_2' = |L(h,i;g+hj)|$.

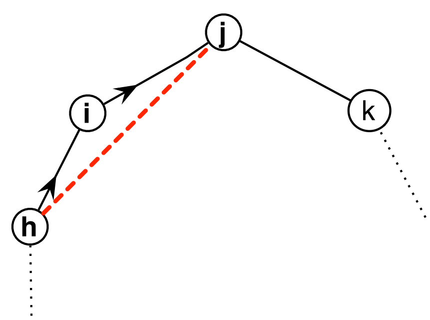

Figure 10: Case II – node h pays for the link to node i.

Similar to before, the sets $L(h,i;g+hj)$ and $L(i,j;g+ik)$ are disjoint. To see this, suppose by contradiction that there exists a $z \in L(h,i;g+hj) \cap L(i,j;g+ik)$. This implies that $hi\ldots z$ is the shortest path in $g+hj$, and $ij\ldots z$ is the shortest path in $g+ik$. Since by definition, a shortest path does not traverse the same node twice, $hi\ldots z$ is still the shortest path in g, and $ij\ldots z$ is still the shortest path in g. Putting these two together, $hij\ldots z$ is the shortest path in g, and hence $hj\ldots z$ is the shortest path in $g+hj$. This contradicts the assumption that $z \in L(h, i; g + hj)$, i.e. all the shortest paths from h to z in $g + hj$ must be of the form $hi \dots z$.

Since the sets $L(h, i; g + hj)$ and $L(i, j; g + ik)$ are disjoint, we must have $n - 4 \geq L_1 + L'_2$, where $L_1 = L(i, j; g + ik)$, and $L'_2 = L(h, i; g + hj)$. Accordingly, the worst case upper bound occurs when $L_1 = L'_2 = \lfloor \frac{n-4}{2} \rfloor$.

Putting the steps together, it takes at most $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil + 1 = \lceil \frac{b(1)+(b(2)-b(3))(L_1+1)-c}{b(2)-b(3)} \rceil$ number of costless links to perturb the Nash equilibrium, where $L_1 = \lfloor \frac{n-4}{2} \rfloor$. Since this number cannot be less than 1, the upper bound is more precisely written as $\max \left\{ \frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2 \rfloor - c}{b(2)-b(3)}, 1 \right\}$, which is the expression stated in Theorem 8 above. This concludes the proof.

□

7. APPENDIX

7.1. PROOF OF PROPOSITION 1

PROOF: (i) For $b(1) - b(2) > c$, the unique Nash equilibrium of the game is the complete network, if the distance between some two agents is more than 1, then forming a link to each other gives a marginal benefit of $b(1) - b(k)$, where $k > 1$. Since $b(1) - b(k) > b(1) - b(2) > c$, either agent would be willing to pay c to form a link.

(ii) For $b(1) - b(2) < c < b(1)$, any star is a Nash equilibrium but there are other equilibria. The star is an equilibrium because if any agent deletes his current link, his utility decreases by at least $b(1) - c$, which is a positive amount. There is no incentive to form additional links because all agents are at most two links apart, and the marginal benefit of any additional link is just $b(1) - b(2)$, which is less than the marginal cost c.

(iii) For $b(1) < c < b(1) + b(2)(n - 2)$, the empty network is a Nash equilibrium. In an empty network, the marginal benefit from any link is $b(1)$, which is less than c, so no agent would form any link. The periphery-sponsored star is a Nash equilibrium because if any of the periphery agent severs the link to the center, he would lose $b(1) + b(2)(n - 2)$ but only gain c. In the PS star, all agents are at most two links apart, hence the marginal benefit of any additional link is just $b(1) - b(2)$, which is less than the marginal cost c. Therefore in a PS star, no agent would prefer to deviate by deleting, forming, or switching link from the center to another periphery node.

(iv) For $c > b(1) + b(2)(n-2)$, the empty network is the unique Nash equilibrium. Since $c > b(1) + b(2)(n-2) \implies c > b(1)$, the result follows from the previous paragraph.

□

7.2. PROOF OF LEMMA 2

PROOF: *If a Nash equilibrium is nonempty, then it must be connected, i.e. there is exactly one component.* Denote the Nash equilibrium strategy profile by s, and the corresponding network by g. Because the equilibrium is nonempty, we can find one component C_1 such that there is a player $i \in C_1$ with $|s_i| \geq 1$. That is, agent i is paying for some links. Now suppose that agent i is linked to agent j, his marginal benefit from the link (i,j) is given by Equation (3) below, and it is positive because g is a Nash equilibrium.

$$\begin{aligned}
&u_i(g) - u_i(g - ij) = \\
&b(1) - b(d(i,j;g-ij)) + \sum_{k \in L(i,j;g)} [b(d(i,k;g)) - b(d(i,k;g-ik))] - c > 0
\end{aligned} \tag{3}$$

Here, $L(i,j;g)$ is the set of nodes for which all the shortest paths from i to $k \in L(i,j;g)$ contains the node j in the undirected network of g.

Suppose by contradiction that the network consists of more than one component, and let us pick an agent i' from this other component C_2. We will now show that $u_{i'}(g + i'j) - u_{i'}(g) > 0$, and hence agent i' has incentive to deviate from the Nash strategy by forming a link to agent j across components.

$$\begin{aligned}
&u_{i'}(g + i'j) - u_{i'}(g) \\
&> b(1) - 0 + \sum_{k \in L(i,j;g) \subset C_1} \left[b(d(i',k;g+i'j)) - 0 \right] - c \\
&> u_i(g) - u_i(g - ij) \qquad (4) \\
&> 0
\end{aligned}$$

The inequality in line (4) follows from line (3), along with the fact that $d(i,k;g) = d(i',k;g+i'j)$ for all $k \in L(i,j;g)$. This contradicts the assumption that the strategy profile is a Nash equilibrium. □

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