

## LECTURE 4: MULTIVARIATE RANDOM VARIABLES II

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### 1. Important identities

#### 1.1. Law of iterated expectations

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$$

Now  $\mathbb{E}[Y|X]$  is a scalar random variable, and inhabits the same probability space as  $X$ . Therefore, the outer expectation on the right-hand side is taken with respect to  $f_X(x)$ .

$$\begin{aligned}\mathbb{E}[Y|X = x] &= \int_{-\infty}^{\infty} y f_{Y|X=x}(y|x) dy \\ &= g(x)\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\mathbb{E}[Y|X]] &= \mathbb{E}[g(X)] \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \right) f(x) dx\end{aligned}$$

Intuitively, suppose we use realizations of the variable  $X$  to predict  $Y$ . Then the average of the predicted values over  $X$  equals to the average of  $Y$ .

#### **Example:**

Recall the pdf  $f(x, y) = x + y$  with the support on  $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ . Previously, we found that:

$$\mathbb{E}[Y|X] = \frac{2 + 3X}{3 + 6X}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|X]] &= \int \frac{2+3x}{3+6x} f_X(x) dx \\
&= \int_0^1 \frac{2+3x}{3+6x} \left(\frac{1}{2} + x\right) dx \\
&= \frac{1}{6} \left(\frac{3x^2}{2} + 2x\right) \Big|_0^1 \\
&= \frac{7}{12} \\
&= \mathbb{E}[Y]
\end{aligned}$$

### 1.2. Important properties of conditional expectations

This section is adapted from Chapter 2 of “Econometric Analysis of Cross Section and Panel Data” by Jeffrey M. Wooldridge.

Let  $Y, W$  be random variables. Let  $X$  be the random variable such that  $X = g(W)$ , for some function  $g$ .

Comparing  $\mathbb{E}[Y|W]$  and  $\mathbb{E}[Y|X]$ , we can think of  $\mathbb{E}[Y|X]$  as conditioning on a set of events that is a subset of the set of events being conditioned on in  $\mathbb{E}[Y|W]$ . Because if we know the outcome of  $W$ , then we would know  $X$ , but the converse is not true.

$$(1) \quad \mathbb{E}[\mathbb{E}[Y|X]|W] = \mathbb{E}[Y|X]$$

$$(2) \quad \mathbb{E}[\mathbb{E}[Y|W]|X] = \mathbb{E}[Y|X]$$

A phrase useful for remembering both equations above: “The smaller information set always dominates”. This is also known as the Tower Property of conditional expectations, which can be demonstrated more formally with measure-theoretic notations.

Some consequences of this useful property:

$$(3) \quad \mathbb{E}[\mathbb{E}[Y|X]|X^2] = \mathbb{E}[\mathbb{E}[Y|X^2]|X] = \mathbb{E}[Y|X^2]$$

$$(4) \quad \mathbb{E}[\mathbb{E}[Y|X, Z]|X] = \mathbb{E}[\mathbb{E}[Y|X]|X, Z] = \mathbb{E}[Y|X]$$

### 1.3. Conditional variance identity

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$$

$\mathbb{E}[Y|X]$  and  $\text{Var}(Y|X)$  are each scalar random variable that is a transformation of  $X$  and has the same probability space as  $X$ . Therefore, the expectation and variance on the right-hand side is taken with respect to the pdf  $f_X(x)$ .

#### Example:

Using the same example as before, we have the pdf  $f(x, y) = x + y$  with the support on  $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ .

$$\begin{aligned} \mathbb{E}[Y|X] &= \frac{2 + 3X}{3 + 6X} \\ \text{Var}(\mathbb{E}[Y|X]) &= \mathbb{E}[(\mathbb{E}[Y|X])^2] - (\mathbb{E}[\mathbb{E}[Y|X]])^2 \\ &= \int_0^1 \left(\frac{2 + 3x}{3 + 6x}\right)^2 f_X(x) dx - \mathbb{E}[Y]^2 \\ &= \int_0^1 \left(\frac{2 + 3x}{3 + 6x}\right)^2 \left(\frac{1}{2} + x\right) dx - \left(\frac{7}{12}\right)^2 \\ &= \frac{1}{288}(96 + \log(9)) - \frac{49}{144} \end{aligned}$$

We can derive  $\text{Var}(Y|X)$  by:

$$\begin{aligned} \text{Var}[Y|X = x] &= \mathbb{E}[Y^2|X = x] - (\mathbb{E}[Y|X = x])^2 \\ &= \int_0^1 y^2 f_{Y|X=x}(y|x) dy - (\mathbb{E}[Y|X = x])^2 \\ &= \int_0^1 y^2 \frac{2(x+y)}{1+2x} dy - \left(\frac{2+3x}{3+6x}\right)^2 \\ &= \frac{4x+3}{12x+6} - \left(\frac{2+3x}{3+6x}\right)^2 \\ &= \frac{1}{36} \left(3 - \frac{1}{(2x+1)^2}\right) \end{aligned}$$

$$\begin{aligned}\mathbb{E}[\text{Var}[Y|X]] &= \int_0^1 \frac{1}{36} \left( 3 - \frac{1}{(2x+1)^2} \right) f_X(x) dx \\ &= \frac{1}{144} (12 - \log(3))\end{aligned}$$

Therefore,  $\mathbb{E}[\text{Var}[Y|X]] + \text{Var}(\mathbb{E}[Y|X]) = \frac{11}{144} = \text{Var}(Y)$ .

## 2. Example: putting everything together

Suppose  $X$  and  $Y$  are distributed uniformly on the triangle  $(0, 0), (0, 1), (1, 0)$ . That is:

$$f_{X,Y}(x, y) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1.) Is this a valid pdf?

$$\int_0^1 \int_0^{1-y} 2 dx dy$$

Performing the inner integral first with respect to  $x$ :

$$\begin{aligned}\int_0^1 [2x]_0^{1-y} dx &= \int_0^1 2(1-y) dy \\ &= 2 \left[ y - \frac{y^2}{2} \right]_0^1 = 2 \left( 1 - \frac{1}{2} \right) = 1\end{aligned}$$

2.) Derive the marginal pdfs.

$$f_X(x) = \int_0^{1-x} 2 dy = 2(1-x) \text{ for } x \in [0, 1]$$

$$f_Y(y) = \int_0^{1-y} 2 dx = 2(1-y) \text{ for } y \in [0, 1]$$

3.) Calculate  $\text{Cov}(X, Y)$

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y]$$

$$\mathbb{E}[X] = \int_0^1 2(1-x)x dx = \frac{1}{3}$$

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 2(1-y)y \, dy = \frac{1}{3} \\
\mathbb{E}[XY] &= \int \int xy f(x, y) \, dx \, dy \\
&= \int_0^1 \int_0^{1-y} 2xy \, dx \, dy \\
&= \int_0^1 [x^2 y]_0^{1-y} \, dy \\
&= \int_0^1 (1-y)^2 y \, dy \\
&= \left[ \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 \\
&= \frac{1}{12}
\end{aligned}$$

Hence  $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y] = \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{36}$

4.) Calculate  $P(Y \leq 1 - 2X)$ :

$$\begin{aligned}
P(Y \leq 1 - 2X) &= \int_0^{1/2} \int_0^{1-2x} f(x, y) \, dy \, dx \\
&= \int_0^{1/2} 2 - 4x \, dx \\
&= [2x - 2x^2]_0^{1/2} \\
&= \frac{1}{2}
\end{aligned}$$

5.) Derive  $\mathbb{E}[Y|X = x]$  and  $\text{Var}(Y|X = x)$ :

First, the density of  $Y|X = x$ :

$$f_{Y|X=x}(y|x) = \frac{f(x, y)}{f(x)} = \frac{2}{2(1-x)}, \text{ for } 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1$$

Conditional expectation:

$$\mathbb{E}(Y|X = x) = \int_0^{1-x} y f_{Y|X=x}(y|x) dy = \int_0^{1-x} \frac{y}{(1-x)} dy = \frac{1-x}{2}$$

Conditional variance:

$$\begin{aligned} \text{Var}(Y|X = x) &= \mathbb{E}[Y^2|X = x] - \mathbb{E}[Y|X = x]^2 \\ &= \int_0^{1-x} y^2 f_{Y|X=x}(y|x) dy - \left(\frac{1-x}{2}\right)^2 \\ &= \frac{1}{3}(1-x)^2 - \left(\frac{1-x}{2}\right)^2 \\ &= \frac{1}{12}(1-x)^2 \end{aligned}$$

6.) Derive  $\text{Var}(\mathbb{E}[Y|X])$  and  $\mathbb{E}[\text{Var}(Y|X)]$ :

$$\begin{aligned} \text{Var}(\mathbb{E}[Y|X]) &= \mathbb{E}[(\mathbb{E}[Y|X])^2] - \mathbb{E}[\mathbb{E}[Y|X]]^2 \\ &= \int_0^1 \left(\frac{1-x}{2}\right)^2 2(1-x) dx - \mathbb{E}[Y]^2 \\ &= \frac{1}{8} - \frac{1}{9} \\ &= \frac{1}{72} \end{aligned}$$

Alternatively,

$$\begin{aligned} \text{Var}(\mathbb{E}[Y|X]) &= \text{Var}\left(\frac{1-X}{2}\right) \\ &= \frac{1}{4}\text{Var}(X) \\ &= \frac{1}{4}\left(\int x^2 2(1-x) dx - \mathbb{E}[X]^2\right) \\ &= \frac{1}{4}\left(\frac{1}{6} - \frac{1}{9}\right) \\ &= \frac{1}{4} \times \frac{1}{18} = \frac{1}{72} \end{aligned}$$

$$\begin{aligned}\mathbb{E}(\text{Var}[Y|X]) &= \int_0^1 \frac{1}{12}(1-x)^2 \cdot 2(1-x) dx \\ &= \frac{1}{24}\end{aligned}$$

Indeed, we see that the Conditional Variance Identity holds true here.  $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$ , where  $\text{Var}(Y) = \int_0^1 y^2 2(1-y) dy - \frac{1}{9} = \frac{1}{18}$ .

### 3. Transformation of bivariate random variables

Let  $(X, Y)$  be a bivariate random vector. Consider a new bivariate random vector  $(U, V)$  defined by  $U = g_1(X, Y)$ ,  $V = g_2(X, Y)$ . What is the probability distribution of  $(U, V)$ ?

Let  $\mathcal{A}$  denote the support of the  $(X, Y)$ , i.e.  $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : f_{X,Y}(x, y) > 0\}$ .

The transformation is  $U = g_1(X, Y)$  and  $V = g_2(X, Y)$ . The support of  $(U, V)$  is then  $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u = g_1(x, y), v = g_2(x, y) \text{ for some } (x, y) \in \mathcal{A}\}$ .

Assume that  $g_1$  and  $g_2$  are functions such that the relationship between  $\mathcal{A}$  and  $\mathcal{B}$  is one-to-one and onto (a bijection). For each  $(u, v) \in \mathcal{B}$ , there is only one  $(x, y) \in \mathcal{A}$  such that  $u = g_1(x, y)$  and  $v = g_2(x, y)$ .

As such, we can solve the equations  $u = g_1(x, y)$  and  $v = g_2(x, y)$  in terms of  $x$  and  $y$ . That is, there is an inverse transformation such that  $x = h_1(u, v)$  and  $y = h_2(u, v)$ , where  $h_1$  and  $h_2$  are differentiable functions.

Define the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix}$$

The determinant of the Jacobian matrix is:

$$J = \det(\mathbf{J}) = \begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{vmatrix}$$

That is,  $J = \frac{\partial h_1}{\partial u} \frac{\partial h_2}{\partial v} - \frac{\partial h_1}{\partial v} \frac{\partial h_2}{\partial u}$ .

The joint pdf of  $(U, V)$  is:

$$f_{U,V}(u, v) = \begin{cases} f_{X,Y}(h_1(u, v), h_2(u, v)) |J| & \text{for } (u, v) \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

$|\det(\mathbf{J})|$  is often called the Jacobian, or the Jacobian of the transformation, or the Jacobian determinant. Note that  $\det(\mathbf{J})$  is a function of  $u, v$ . Moreover,  $\det(\mathbf{J}) \neq 0$  since there is an inverse transformation such that  $x = h_1(u, v)$  and  $y = h_2(u, v)$ , where  $h_1$  and  $h_2$  are differentiable functions. The Jacobian is also used during change-of-variables in multiple integrals.

### 3.1. Example

Let  $X$  and  $Y$  be independent, standard Normal random variables.

Consider the transformation  $U = X + Y$  and  $V = X - Y$ . What is the joint pdf of  $(U, V)$ ?

The joint pdf of  $(X, Y)$  is just  $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}$  since  $X$  and  $Y$  are independent.

The support of  $(X, Y)$  is  $\mathbb{R}^2$ . It follows that  $U$  and  $V$  can also take any value from  $-\infty$  to  $\infty$ .

The inverse transformation is  $x = h_1(u, v) = \frac{u+v}{2}$  and  $y = h_2(u, v) = \frac{u-v}{2}$ .

The Jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Hence the joint pdf of  $(U, V)$  is:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(h_1(u, v), h_2(u, v)) |J| \\ &= \frac{1}{2\pi} e^{-\frac{(u+v)^2}{2}} e^{-\frac{(u-v)^2}{2}} \frac{1}{2} \\ &= \left( \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{u^2}{4}} \right) \left( \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{v^2}{4}} \right) \end{aligned}$$

Note that the pdf of  $N(\mu, \sigma^2)$  is  $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ .



Hence the joint pdf of  $(U, V)$  can be factored into two functions  $f_U(u)$  and  $f_V(v)$ . Moreover,  $f_U(u)$  is the pdf of  $N(0, 2)$ . That is,  $U \sim N(0, 2)$  and  $V \sim N(0, 2)$ . The sum ( $U$ ) and difference ( $V$ ) of independent normal random variables are independent normal random variables, as long as  $\text{Var}(X) = \text{Var}(Y)$ .

We can also consider the ratio and the product of Normal variables. Consider the transformation  $U = X/Y$  and  $V = X$ . What is the joint pdf of  $(U, V)$ ? What about the product  $V = XY$ ?

### 3.2. Discrete bivariate random vectors

Let  $(X, Y)$  be a discrete bivariate random vector. Let  $\mathcal{A}$  be the support of  $(X, Y)$ , i.e. the set of points where the joint pmf of  $(X, Y)$  takes strictly positive values. Note that  $\mathcal{A}$  must be a countable set (either finite or countably infinite).

The joint pmf of  $(U, V)$  is:

$$f_{U,V}(u, v) = P(U = u, V = v) = \sum_{(x,y) \in \mathcal{A}: g_1(x,y)=u, g_2(x,y)=v} f_{X,Y}(x, y)$$

## 4. Some important inequalities

### 4.1. Jensen's Inequality

A function  $g(x)$  is convex if and only if  $\lambda g(x) + (1 - \lambda)g(y) \geq g(\lambda x + (1 - \lambda)y)$  for  $0 < \lambda < 1$ . Graphically, a straight line connecting any two points of the convex function lies above the function.

**Jensen's Inequality:** For any random variable  $X$ , if  $g(X)$  is convex, then  $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$ .

For example: take  $g(X) = X^2$ , then  $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ , which implies that  $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 \geq 0$ .

### 4.2. Concentration inequalities (Markov and Chebyshev's Inequalities)

Concentration inequalities provide bounds on the probabilities of a random variable deviating from a certain value. Markov's inequality and Chebyshev's inequality are examples of concentration inequalities. Let  $X$  be a random variable and  $g(X)$  be a non-negative function. Chebyshev's inequality: for any  $\epsilon > 0$ ,

$$P(g(X) \geq \epsilon) \leq \frac{\mathbb{E}[g(X)]}{\epsilon}$$

Proof:

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_{-\infty}^{\infty} g(x)f(x) dx \\
&\geq \int_{x:g(x)\geq\epsilon}^{\infty} g(x)f(x) dx \\
&\geq \int_{x:g(x)\geq\epsilon}^{\infty} \epsilon f(x) dx \\
&= \epsilon P(g(X) \geq \epsilon)
\end{aligned}$$

Markov's inequality is just  $P(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}$ .

Now let  $g(x) = \frac{(x-\mu)^2}{\sigma^2} \geq 0$ , where  $\mu = \mathbb{E}[X]$  and  $\sigma^2 = \text{Var}(X)$ . Note that  $g$  is always positive. By the Chebyshev's inequality,

$$\begin{aligned}
P(g(X) \geq \epsilon^2) &\leq \frac{\mathbb{E}[g(X)]}{\epsilon^2} \\
P\left(\frac{(X-\mu)^2}{\sigma^2} \geq \epsilon^2\right) &\leq \frac{\mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right]}{\epsilon^2} \\
P\left(\frac{(X-\mu)^2}{\sigma^2} \geq \epsilon^2\right) &\leq \frac{1}{\epsilon^2} \\
(5) \quad P(|X - \mu| \geq \epsilon\sigma) &\leq \frac{1}{\epsilon^2}
\end{aligned}$$

If we take  $\epsilon = 2$ , then  $P(|x - \mu| \geq 2\sigma) \leq 0.25$  or  $P(|x - \mu| < 2\sigma) > 0.75$ . That is, there is at least 75% chance that a random variable (any random variable!) will be within 2 standard deviation of its mean.

In general, the Chebyshev's inequality can be used to show that as  $\text{Var}(X_n) \rightarrow 0$ ,  $P(|X_n - \mu| \geq \epsilon) \rightarrow 0$ , by taking  $g(X) = (X - \mu)^2$ .

As such, Chebyshev's inequality can be used to prove the Weak Law of Large Numbers. Let  $X_1, \dots, X_n$  be  $n$  independent random variables, each with the same density  $f$ . Define the sample mean as the random variable  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ . Note that  $\bar{X}$  has expectation  $\mathbb{E}[\bar{X}] \equiv \mu$ , and variance  $\frac{\text{Var}(X)}{n} \equiv \frac{\sigma^2}{n}$ .

By the inequality in (5), we have:

$$P(|\bar{X} - \mu| \geq \epsilon \frac{\sigma}{\sqrt{n}}) \leq \frac{1}{\epsilon^2}$$

Now if we let  $\epsilon = v \frac{\sqrt{n}}{\sigma}$ ,

$$P(|\bar{X} - \mu| \geq v) \leq \frac{\sigma^2}{nv^2}$$

Therefore, as  $n \rightarrow \infty$ ,  $P(|\bar{X} - \mu| \geq v) = 0$  for any  $v > 0$ , which is the Weak Law of Large Numbers.

## 5. Common families of statistical distributions

### 5.1. Multivariate Normal

We are already familiar with the one-dimensional Gaussian random variable  $X \sim \mathcal{N}(\mu, \sigma^2)$ , which has the pdf  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}$  with the support over the entire real line.

The  $k$ -dimensional Gaussian random variable is described as:

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

$\mathbf{X}$  is a  $k$ -dimensional random vector.  $\boldsymbol{\mu}$  is a  $k$ -dimensional vector,  $\Sigma$  is a  $k$ -by- $k$  symmetric matrix called the variance-covariance matrix. A matrix  $\Sigma$  is symmetric if  $\Sigma^T = \Sigma$ , as such  $\Sigma$  has  $k + (k^2 - k)/2 = (k^2 + k)/2$  number of parameters. Intuitively,  $k$  diagonal terms of  $\Sigma$  describe the variances of each individual random variable, and  $(k^2 - k)/2$  off-diagonal terms of  $\Sigma$  describe the pairwise correlations between each of the variable.<sup>1</sup>

Therefore a  $k$ -dimensional Gaussian variable has  $\frac{3k+k^2}{2}$  number of parameters. For example, a 2-dimensional multivariate Gaussian has 5 parameters.

For the bivariate Normal distribution:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right]$$

The pdf of  $(X, Y)$  is:

$$(6) \quad f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left( -\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right] \right)$$

for  $x, y \in \mathbb{R}^2$ . Check that the marginal pdf of  $X$  is just the univariate Normal pdf:

<sup>1</sup>In addition  $\Sigma$  also has to be positive semi-definite, that is,  $\mathbf{x}^T \Sigma \mathbf{x} \geq 0$  for all  $\mathbf{x} \in \mathbb{R}^k$ .

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}, \quad x \in \mathbb{R}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}, \quad y \in \mathbb{R}$$

Hence, the moments of  $(X, Y)$  are described by the parameters of the pdf, i.e.  $\mathbb{E}[X] = \mu_X$ ,  $\mathbb{E}[Y] = \mu_Y$ ,  $\text{Var}(X) = \sigma_X^2$ ,  $\text{Var}(Y) = \sigma_Y^2$ .

In addition, we can compute  $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$  from the joint pdf, which turns out to be  $\rho\sigma_X\sigma_Y$ . As such the correlation of  $X$  and  $Y$  is just  $\rho$ .

If we set  $\rho = 0$ , i.e. zero correlation between  $X$  and  $Y$ , then:

$$f(x, y) = f_X(x)f_Y(y)$$

Hence, for Multivariate Normals, zero correlation implies independence. Also, if  $X$  and  $Y$  are independent with univariate Normal distributions, then  $(X, Y)$  trivially has a bivariate Normal distribution.

However in general, if two random variables  $X$  and  $Y$  are univariate Normals, it is not true that  $(X, Y)$  has a bivariate Normal distribution. Can you work out an example?

The conditional distribution of  $Y$  given  $X = x$  is:

$$(7) \quad (Y|X = x) \sim \mathcal{N}\left(\mathbb{E}[Y] + \rho\frac{\sigma_Y}{\sigma_X}(x - \mathbb{E}[X]), (1 - \rho^2)\sigma_Y^2\right)$$

This implies that the conditional expectation of  $Y$  given  $X$  is:

$$\mathbb{E}[Y|X] = \mathbb{E}[Y] + \rho\frac{\sigma_Y}{\sigma_X}(X - \mathbb{E}[X])$$

It is a **linear** function of  $X$  and has a normal pdf. The fact that  $\mathbb{E}[Y|X]$  is linear in  $X$  means that the best prediction of  $Y$  using  $X$  is some linear function of  $X$ . That is, we can't do better than a linear regression of  $Y$  on  $X$  if  $(Y, X)$  is a bivariate Normal.

The conditional variance of  $Y$  given  $X$  is  $\text{Var}[Y|X] = (1 - \rho^2)\sigma_Y^2$ , which does not depend on  $X$ .

In general, the joint density of a  $k$ -th dimensional multivariate Normal distribution is:

$$f_{\mathbf{X}}(x_1, \dots, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}}$$

Where  $\boldsymbol{\Sigma}$  is a  $k$ -by- $k$  variance-covariance matrix of  $\mathbf{X}$ , and  $\boldsymbol{\mu}$  is a  $k$ -dimensional vector. We say that  $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

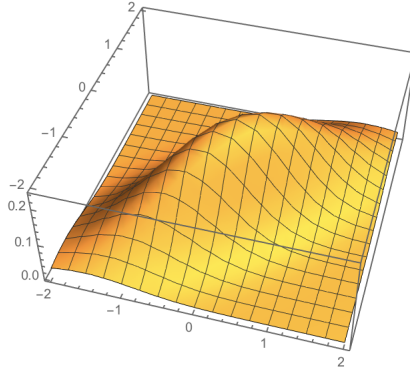
### 5.2. Example

For example, let  $\mu_X = \mu_Y = 0$  and  $\sigma_X = \sigma_Y = 1$  in the joint pdf of Bivariate Normal (Equation 8). The location parameters  $\mu_X$  and  $\mu_Y$  merely shift the center of the distribution around. Then we have:

$$(8) \quad f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right)$$

Visualize this joint pdf at various values of  $\rho$  as in Figure 1.

```
Plot3D[ReplaceAll[ $\frac{1}{2\pi\sqrt{1-\rho^2}} \text{Exp}\left[-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right]$ ,  $\rho \rightarrow 0.75$ ], {x, -2, 2},
{y, -2, 2}]
```



```
Plot3D[ReplaceAll[ $\frac{1}{2\pi\sqrt{1-\rho^2}} \text{Exp}\left[-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right]$ ,  $\rho \rightarrow 0$ ], {x, -2, 2}, {y, -2, 2}]
```

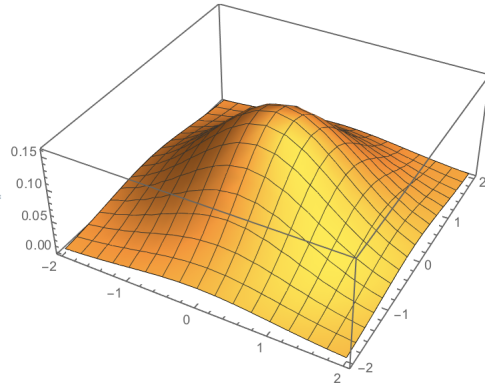


FIGURE 1

Now we derive the conditional distribution of  $X$  given  $Y$ .

$$\begin{aligned}
f_{X|Y=y}(x|y) &= \frac{f(x, y)}{f(y)} \\
&= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right)}{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)} + \frac{y^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+\rho^2 y^2-2\rho xy}{2(1-\rho^2)}\right) \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)
\end{aligned}$$

The last line is the pdf of a univariate Normal distribution with mean  $\rho y$  and variance  $1 - \rho^2$ . Therefore,

$$(X|Y = y) \sim N(\rho y, 1 - \rho^2)$$

Let's also check whether the joint pdf integrates to the marginal pdfs (which can be evaluated analytically by completing the squares):

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right) dx \\
&= \frac{\exp\left(\frac{-y^2}{2(1-\rho^2)}\right)}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2-2\rho xy}{2(1-\rho^2)}\right) dx \\
&= \frac{\exp\left(\frac{\rho^2 y^2 - y^2}{2(1-\rho^2)}\right)}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right) dx \\
&= \frac{\exp\left(\frac{-y^2}{2}\right)}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}
\end{aligned}$$

### 5.3. Sampling from a multivariate Normal

To sample from a scalar random variable, we learned how to use the probability integral transform. We can use the conditional distribution to sample from a multivariate distribution. For instance, to sample from a bivariate Normal distribution:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right]$$

First, we sample from the marginal of  $X$ , which is just  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ .

Recall the conditional distribution of  $Y$  given  $X = x$  is:

$$(Y|X = x) \sim \mathcal{N} \left( \mathbb{E}[Y] + \rho \frac{\sigma_Y}{\sigma_X} (x - \mathbb{E}[X]), (1 - \rho^2) \sigma_Y^2 \right)$$

For every draw of  $x_i$  from the marginal distribution  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ , we then sample  $y_i$  from  $Y|X = x_i$ . The sample  $(x_i, y_i)_{i=1}^n$  will be a valid sample from the bivariate Normal distribution.

This approach is called *Gibbs Sampling*.<sup>2</sup> More generally, to sample from a trivariate distribution  $f(x, y, z)$ , we first draw  $x_i$  from the marginal of  $X$ , then draw  $y_i$  from  $Y|X = x_i$ , then finally, draw  $z_i$  from  $Z|Y = y_i, X = x_i$ . Now, the density of  $Z|Y, X$  can be derived as  $f(x, y, z)/f(x, y)$ .

Let's try to implement Gibbs sampling using R or Python.

### 5.4. Beta distribution

Beta distribution is used to model random variables that lie within the unit interval  $[0, 1]$ . For example, if we want to model fractions or probabilities, then we use the Beta distribution.

The Beta distribution is controlled by two parameters  $\alpha > 0$  and  $\beta > 0$ , that is,  $X \sim \text{Beta}(\alpha, \beta)$ .

The pdf is  $f_X(x) \propto x^{\alpha-1}(1-x)^{\beta-1}$  for  $x \in [0, 1]$ . The constant of proportionality is  $\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ , where  $\Gamma$  is the Gamma function.<sup>3</sup>

<sup>2</sup>More specifically, this is the Collapsed Gibbs Sampling

<sup>3</sup>The Gamma function is an interesting function. It is defined as  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ . The Gamma function satisfies the following recurrence relation:  $\Gamma(z) = (z-1)\Gamma(z-1)$ . As such, when  $z$  is an integer,  $\Gamma(z) = (z-1)!$ . We can think of the Gamma function as an extension of the factorial function to non-negative real numbers. For non-integers  $z > 1$ , it must be that  $\Gamma(z) = (z-1)(z-2)\dots\delta\Gamma(\delta)$  where  $0 < \delta < 1$ .



The Beta distribution is a very flexible class of distributions that can generate distributions that are positively or negatively skewed, varying modes and medians. The mean is given by  $\frac{\alpha}{\alpha+\beta}$ .

The Dirichlet distribution generalizes the Beta distribution to multiple dimensions:

$$f(x_1, \dots, x_K; \alpha_1, \dots, \alpha_K) \propto \prod_{i=1}^K x_i^{\alpha_i-1}$$

Where  $\{x_k\}_{k=1}^{k=K}$  belong to the standard  $K-1$  simplex, or in other words:  $\sum_{i=1}^K x_i = 1$  and  $x_i \geq 0$  for all  $i \in \{1, \dots, K\}$ . The normalizing constant is the multivariate beta function, which can be expressed in terms of the gamma function

$$B(\boldsymbol{\alpha}) = \frac{\prod_{i=1}^K \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^K \alpha_i\right)}, \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_K)$$

### 5.5. Gamma distribution

The Gamma distribution is used to model random variables that takes positive values. It is a general form of the Exponential distribution. It is also used in Bayesian statistics as conjugate priors. Moreover, it is used in the frequentist setting for hypothesis testing.

Let  $X_1, X_2, \dots, X_n$  be  $n$  independent Exponential distribution with parameter  $\lambda$ . Then,  $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda)$ . Therefore the Gamma distribution gives the duration it takes until  $n$  number of event occurrences, where the rate of an event occurrence is  $\lambda$ .

More generally, the Gamma distribution is a two-parameter distribution.  $X \sim \text{Gamma}(\alpha, \beta)$  where  $X$  takes only positive real values and  $\alpha, \beta > 0$ . The pdf is given by  $f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-x\beta}}{\Gamma(\alpha)}$  for  $x \geq 0$ .

If  $X \sim \text{Gamma}(1, \lambda)$ , then  $X$  has an exponential distribution with mean  $\frac{1}{\lambda}$ . If  $X \sim \text{Gamma}(v/2, 1/2)$ , then  $X$  is identical to  $\chi(v)$ , the chi-squared distribution with  $v$  degrees of freedom.

### 5.6. Bernoulli and Binomial Distribution

$X$  is a Bernoulli distribution with parameter  $p$  if  $X = 1$  with probability  $p$ , and  $X = 0$  with probability  $1 - p$ .

Let  $X_1, X_2, \dots, X_n$  be  $n$  independent Bernoulli random variables with parameter  $p$ .  $Y = \sum_{i=1}^n X_i$  is a Binomial distribution with parameters  $(n, p)$ .

$$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$Y$  is the number of successes in  $n$  independent trials, where  $p$  is the probability of a success in a trial. The mean of  $Y$  is  $np$ , and the variance of  $Y$  is  $np(1-p)$ , can you prove this?

### 6. A note on truncated random variables

Consider a random variable  $X$  with density  $f_X(x)$ . What is  $\mathbb{E}[X|X > a]$ ?  $X > a$  is an event, not a random variable, so do not confuse with the formula for deriving conditional density. The density of  $X|X > a$  is  $\frac{1}{1-F_X(a)} f_X(x) \mathbb{1}(x > a)$  with the support truncated to  $x > a$ . Note this density integrates to one.

In general, the density of  $X|X \in (a, b)$  is  $\frac{1}{F_X(b)-F_X(a)} f_X(x) \mathbb{1}(x \in (a, b))$  with the support truncated to  $x \in (a, b)$ . Note this density integrates to one as well.

Now let  $X \sim U[0, 1]$ , and  $a \in (0, 1)$ , what is  $\mathbb{E}[X|X > a]$ ?

$$\begin{aligned} \mathbb{E}[X|X > a] &= \frac{\mathbb{E}[X \mathbb{1}_{\{X > a\}}]}{1 - F_X(a)} \\ &= \frac{\int_a^\infty x f_X(x) dx}{1 - F_X(a)} \\ &= \frac{\int_a^1 x dx}{1 - a} \\ &= \frac{\int_a^1 x dx}{1 - a} \\ &= \frac{a+1}{2} \quad \text{for } a \in (0, 1) \end{aligned}$$

For instance, if  $X \sim \mathcal{N}(0, \sigma^2)$ , then we can use the above formula to show that  $\mathbb{E}[X|X > 0] \approx 0.7978\sigma$ .

Now consider the random variables  $(X, Y)$  which are joint uniformly distributed on the unit square. That is,  $f(x, y) = 1$  for  $0 < x < 1$  and  $0 < y < 1$ . Show that  $\mathbb{E}[X|Y > X] = \frac{1}{3}$ . Note that  $Y > X$  is an event, not a random variable. As such, the formula to compute this conditional expectation is  $\mathbb{E}[X|Y >$

$X] = \frac{\mathbb{E}[X\mathbb{1}_{\{Y>X\}}]}{P(Y>X)}$ , and *not*  $\mathbb{E}[Y|X = x] = \int y \frac{f(x,y)}{f(x)} dy$ , which is the formula when conditioning on a random variable. In general, the density of  $X|(X,Y) \in A$  is  $\int_{-\infty}^{\infty} \frac{1}{\Pr((X,Y) \in A)} f_{X,Y}(x,y) \mathbb{1}((x,y) \in A) dy$ .

$$\begin{aligned}\mathbb{E}[X|Y > X] &= \frac{\mathbb{E}[X\mathbb{1}_{\{Y>X\}}]}{\Pr(Y > X)} \\ &= \frac{\int_0^1 x \int_x^1 f_{X,Y}(x,y) dy dx}{1/2} \\ &= \frac{\int_0^1 x(1-x) dx}{1/2} \\ &= \frac{1}{3}\end{aligned}$$