

## LECTURE 3: MULTIVARIATE RANDOM VARIABLES

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INSTRUCTOR: DR. KHAI CHIONG  
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Previously, we looked at univariate random variables, that is, the variable of interest is a scalar. Most of the time however, we are interested in the behavior of a vector. For instance, the behavior of (i) quantities and prices, (ii) employment and GDP, (iii) customer shopping frequency and spending, (iv) temperature and rainfall, (v) prices of multiple assets etc.

An  $n$ -dimensional random vector is a function from a sample space  $\Omega$  into  $\mathbb{R}^n$ , the  $n$ -dimensional Euclidean space.

### 1. Pdf and pmf of bivariate random variables

#### 1.1. Discrete case

Consider the experiment of tossing two fair dice. The sample space of this experiment is the set of all the possible outcomes.  $\Omega = \{(1, 1), (1, 2), \dots, (2, 1), \dots\}$ , where  $|\Omega| = 36$ .

Define  $X =$  sum of the two dice,  $Y =$  |difference of the two dice|. In this way, we have defined the bivariate random vector  $(X, Y)$ .

1.) What is  $P(X = 6, Y = 0)$ ? The event  $X = 6$  and  $Y = 0$  occurs if and only if the two dice are 3. Hence,  $P(X = 6, Y = 0) = \frac{1}{36}$ .

2.) How about  $P(X = 8, Y = 2)$ ?

$$P(X = 8, Y = 2) = \frac{1}{18}$$

3.) How about  $P(X = 7, Y \leq 4)$ ?

$$P(X = 7, Y \leq 4) = \sum_{y=0}^4 P(X = 7, Y = y) = \frac{4}{36} = \frac{1}{9}$$

First die, second die	1	2	3	4	5	6
1	(2, 0)	(3, 1)	(4, 2)	(5, 3)	(6, 4)	(7, 5)
2	(3, 1)	(4, 0)	(5, 1)	(6, 2)	(7, 3)	(8, 4)
3	(4, 2)	(5, 1)	(6, 0)	(7, 1)	(8, 2)	(9, 3)
4	(5, 3)	(6, 2)	(7, 1)	(8, 0)	(9, 1)	(10, 2)
5	(6, 4)	(7, 3)	(8, 2)	(9, 1)	(10, 0)	(11, 1)
6	(7, 5)	(8, 4)	(9, 3)	(10, 2)	(11, 1)	(12, 0)

TABLE 1. All possible outcomes of  $(X, Y)$ . Each of the realization per cell is equally likely.

Let  $(X, Y)$  be a discrete bivariate random vector. Then the function  $f(x, y)$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  defined by  $f(x, y) = P(X = x, Y = y)$  is called the joint probability mass function of  $(X, Y)$ . The notation  $f_{X,Y}(x, y)$  will also be used.

### 1.1.1. Marginal pmf

Given the joint pmf  $f_{X,Y}(x, y)$ , the marginal pmf of  $X$  denoted by  $f_X(x)$  is given by:

$$f_X(x) = \sum_{y \in \mathbb{R}} f_{X,Y}(x, y)$$

Similarly, the marginal pmf of  $Y$  denoted by  $f_Y(y)$  is given by:

$$f_Y(y) = \sum_{x \in \mathbb{R}} f_{X,Y}(x, y)$$

Consider the dice experiment above, what is  $f_X(3) = P(X = 3)$ ?

$$\begin{aligned} f_X(3) &= \sum_{y \in \mathbb{R}} f_{X,Y}(3, y) \\ &= \sum_{y=0}^5 P(X = 3, Y = y) \\ &= P(X = 3, Y = 1) \\ &= \frac{1}{18} \end{aligned}$$

**1.2. Continuous case**

A function  $f(x, y)$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  is called a joint probability density function or joint pdf of the continuous bivariate random vector  $(X, Y)$  if for every  $A \subseteq \mathbb{R}^2$ :

$$P((X, Y) \in A) = \int \int_A f_{X,Y}(x, y) dx dy$$

Any function  $f(x, y)$  satisfying  $f(x, y) \geq 0$  for all  $(x, y) \in \mathbb{R}^2$  and

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy$$

is the joint pdf of some continuous bivariate random vector  $(X, Y)$ .

**Example:** consider the following function.

$$f(x, y) = \begin{cases} 6xy^2 & 0 < x < 1, \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The support of  $(X, Y)$  is the unit square. We check that  $P((X, Y) \in \mathbb{R}^2) = 1$ .

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_0^1 \int_0^1 6xy^2 dx dy \\ &= \int_0^1 3y^2 dy \\ &= 1 \end{aligned}$$

What is  $P((X, Y) \in A)$ , where  $A$  is the region defined by  $A = \{(x, y) \in \mathbb{R}^2 : x \leq \frac{1}{2}, y \leq \frac{1}{2}\}$ ?

$$\begin{aligned}
P((X, Y) \in A) &= P\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) \\
&= \int_{-\infty}^{1/2} \int_{-\infty}^{1/2} f(x, y) \, dx \, dy \\
&= \int_0^{1/2} \int_0^{1/2} 6xy^2 \, dx \, dy \\
&= \int_0^{1/2} \frac{3}{4}y^2 \, dy \\
&= \frac{1}{32}
\end{aligned}$$

We can visualize the joint pdf using Mathematica. We will see that geometric intuitions can be useful sometimes – we interpret  $P((X, Y) \in A)$  as the volume underneath the curve  $f(x, y)$  with respect to the region  $A$ .

**Example:** consider again the pdf  $f(x, y) = 6xy^2$  with the support on the unit square. What is  $P(X + Y \geq 1)$ ?

Let  $A$  be the region in 2-dimensional Euclidean space such that  $A = \{(x, y) \in \mathbb{R}^2 : x + y \geq 1, 0 < x < 1, 0 < y < 1\}$ . Essentially we are asking  $P((X, Y) \in A)$ . Graphically,  $A$  is the upper-right triangle of the unit square.

$$\begin{aligned}
A &= \{(x, y) \in \mathbb{R}^2 : x + y \geq 1, 0 < x < 1, 0 < y < 1\} \\
&= \{(x, y) \in \mathbb{R}^2 : x \geq 1 - y, 0 < x < 1, 0 < y < 1\} \\
&= \{(x, y) \in \mathbb{R}^2 : 1 - y \leq x < 1, 0 < y < 1\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(X + Y \geq 1) &= \int \int_A f(x, y) dx dy = \int_0^1 \int_{1-y}^1 6xy^2 dx dy \\
&= \int_0^1 [3x^2y^2]_{1-y}^1 dy \\
&= \int_0^1 3y^2 - 3(1-y)^2y^2 dy \\
&= \int_0^1 3y^2 - 3y^2 + 6y^3 - 3y^4 dy \\
&= \left[ \frac{3}{2}y^4 - \frac{3}{5}y^5 \right]_0^1 \\
&= \frac{9}{10}
\end{aligned}$$

**Example:** consider the following function.

$$f(x, y) = \begin{cases} 1 & 0 < x < 1, \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

This volume of this pdf is just the unit cube. Calculate  $P(X^2 + Y^2 \leq 1)$ . First, we show using brute-force algebra that  $P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}$ , then we use a simple geometric argument that  $P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}$ .

$P(X^2 + Y^2 \leq 1)$  equals to  $P((X, Y) \in A)$  where  $A = \{X^2 + Y^2 \leq 1\}$ .

$$\begin{aligned}
P(X^2 + Y^2 \leq 1) &= \int \int_{\{(x,y):x^2+y^2 \leq 1\}} f(x, y) dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx dy \\
&= \int_0^1 \int_0^{\sqrt{1-y^2}} 1 dx dy \\
&= \int_0^1 \sqrt{1-y^2} dy \\
&= \left[ \frac{1}{2} \left( y\sqrt{1-y^2} + \sin^{-1}(y) \right) \right]_0^1 \\
&= \frac{\pi}{4}
\end{aligned}$$

However, because the pdf has a uniform height of one with the support on the unit square,  $P(X^2 + Y^2 \leq 1)$  is just the volume of a cylinder split into 4 equal parts. Specifically, this cylinder has a height of one, and a radius of one.

### 1.2.1. Marginal pdf

The marginal pdf of  $X$  is defined as:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad \text{for } x \in \mathbb{R}$$

The marginal pdf of  $Y$  is defined as:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx, \quad \text{for } y \in \mathbb{R}$$

**Example:** consider again the pdf  $f(x, y) = 6xy^2$  with the support on the unit square.

Derive the marginal pdf of  $X$ . Then, calculate  $P(\frac{1}{2} < X < \frac{3}{4})$ .

$$\begin{aligned} f_X(x) &= \int_0^1 6xy^2 dy = 2x \\ P\left(\frac{1}{2} < X < \frac{3}{4}\right) &= \int_{\frac{1}{2}}^{\frac{3}{4}} f_X(x) dx \\ &= \int_{\frac{1}{2}}^{\frac{3}{4}} 2x dx \\ &= \frac{5}{16} \end{aligned}$$

## 2. Joint cdf

The bivariate cdf of  $(X, Y)$  is defined as:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

When  $(X, Y)$  is a continuous random vector, then

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(s, t) ds dt$$

From the fundamental theorem of calculus, this implies that

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

The marginal cdf  $F_X(x)$  can be obtained from  $\lim_{y \rightarrow \infty} F(x, y) = F_X(x)$ .

**Example:** consider the cdf:

$$F(x, y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ x & 0 \leq x \leq 1, y > 1 \\ y & 0 \leq y \leq 1, x > 1 \\ 1 & x > 1, y > 1 \end{cases}$$

Therefore by calculating  $f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$ :

$$f(x, y) = \begin{cases} 0 & \text{otherwise} \\ 1 & 0 \leq x \leq 1, 0 \leq y \leq 1 \end{cases}$$

Also check that the marginal cdf  $F_X(x)$  can be obtained from  $\lim_{y \rightarrow \infty} F(x, y) = F_X(x) = x$ .

### 3. Expectation

Let  $g$  be a function from  $\mathbb{R}^2$  to  $\mathbb{R}$ . For the discrete case,

$$\mathbb{E}[g(X, Y)] = \sum_{(x, y) \in \mathbb{R}^2} g(x, y) P(X = x, Y = y)$$

Take  $g(X, Y) = XY$ . What is  $\mathbb{E}[XY]$  in the dice experiment above?

$$\mathbb{E}[XY] = g(1, 0)P(X = 1, Y = 0) + g(1, 1)P(X = 1, Y = 1) + \dots$$

For the continuous case, we have:

$$\mathbb{E}[g(X, Y)] = \int \int_{(x,y) \in \mathbb{R}^2} g(x, y) f_{X,Y}(x, y) dx dy$$

**Example:**

Throw darts randomly at a unit square, record the  $x$ -coordinates and  $y$ -coordinates, and multiply them together. On average, what value would you expect?

In another words, consider the pdf  $f(x, y) = 1$  with the support on the unit square. What is  $\mathbb{E}[XY]$ ?

$$\begin{aligned} \mathbb{E}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dx dy \\ &= \int_0^1 \int_0^1 xy dx dy \\ &= \int_0^1 y/2 dy \\ &= \frac{1}{4} \end{aligned}$$

What about  $\mathbb{E}[X]$  or  $\mathbb{E}[Y]$ ? Calculate the marginals first.

**Example:** What if we don't throw darts uniformly but try to aim away from the origin? Consider again the pdf  $f(x, y) = 6xy^2$  with the support on the unit square.

$$\begin{aligned} \mathbb{E}[XY] &= \int_0^1 \int_0^1 xy 6xy^2 dx dy \\ &= \int_0^1 2y^3 dy \\ &= \frac{1}{2} \end{aligned}$$

How would we calculate  $\mathbb{E}[X^2Y]$ ?

$$\begin{aligned} \mathbb{E}[X^2Y] &= \int_0^1 \int_0^1 x^2y 6xy^2 dx dy \\ &= \frac{3}{8} \end{aligned}$$



#### 4. Conditional probabilities

Consider the bivariate random variables  $(X, Y)$ . The random variable  $Y$  conditional on  $X = x$  is denoted by  $Y|X = x$ . Now,  $Y|X = x$  is another random variable, but it is a (*scalar*) random variable. The density of  $Y|X = x$  is given by:

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

**Example:**

Consider the random variables  $(X, Y)$  that has the joint pdf  $f(x, y) = 6xy^2$  for  $0 < x < 1$  and  $0 < y < 1$ . Consider the random variable  $Y|X = 0.5$ . This random variable is a scalar random variable. The pdf of  $Y|X = 0.5$  is in terms of  $y$  only:

$$\begin{aligned} f_{Y|X=0.5}(y) &= \frac{f_{X,Y}(0.5, y)}{f_X(0.5)} \\ &= \frac{6(0.5)y^2}{2(0.5)} \\ &= 3y^2 \quad \text{for } 0 < y < 1 \end{aligned}$$

Now consider  $Y|X$ , which is a *bivariate* random variable, unlike  $Y|X = x$ , which is a scalar random variable. In particular, the joint density of  $Y|X$  is:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

The two pdfs look identical, except in the pdf of  $Y|X = x$ , we treat  $x$  as fixed and so  $f_{Y|X=x}(y|x)$  is a one-dimensional function. While in the pdf of  $Y|X$ , we treat  $x$  to be an argument of the function, so that the pdf of  $Y|X$  is two-dimensional. That is,  $f_{Y|X=x} : \mathbb{R} \rightarrow \mathbb{R}$ , but  $f_{Y|X} : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

**Example:**

Consider again the bivariate random variable  $(X, Y)$  that has the joint pdf  $f(x, y) = 6xy^2$  for  $0 < x < 1$  and  $0 < y < 1$ . The joint density of  $Y|X$  is given by:

$$\begin{aligned}
 f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
 &= \frac{6xy^2}{2x} \\
 &= \begin{cases} 3y^2 & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Note:  $f_{Y|X} : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function of both  $x$  and  $y$ . Here, the support of the function explicitly depends on  $x$ .

**Example:** Consider the joint density  $f(x, y) = x+y$ , with support on  $(x, y) \in [0, 1]^2$ . What is the joint pdf of  $Y|X$ ?

First show that the marginal density of  $X$  is  $f_X(x) = \frac{1}{2} + x$ , for  $x \in [0, 1]$ . Therefore the conditional density is:

$$f_{Y|X}(y|x) = \begin{cases} \frac{2(x+y)}{1+2x} & (x, y) \in [0, 1]^2 \\ 0 & \text{otherwise} \end{cases}$$

We can also check that the density of  $Y|X = 0$  is  $f_{Y|X=0}(y) = 2y$  for  $y \in [0, 1]$ .

#### 4.1. Conditional expectation

Consider the random variable  $Y|X = x$ . The expectation  $\mathbb{E}[Y|X = x]$  is defined as  $\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X=x}(y|x) dy$ . Note that  $\mathbb{E}[Y|X = x]$  is a constant. In general, we have  $\mathbb{E}[g(Y)|X = x] = \int_{-\infty}^{\infty} g(y) f_{Y|X=x}(y|x) dy$ , for some function  $g$ .

**Example:** consider again the joint pdf  $f(x, y) = 6xy^2$  for  $0 < x < 1$  and  $0 < y < 1$ . The density of  $Y|X = x$  is  $f_{Y|X=x}(y|x) = 3y^2$  for  $0 < y < 1$ . What is  $\mathbb{E}[X|Y = y]$ ?

$$\begin{aligned}
 \mathbb{E}[Y|X = x] &= \int_{-\infty}^{\infty} y f_{Y|X=x}(y|x) dy \\
 &= \int_0^1 3y^3 dy \\
 &= \frac{3}{4}
 \end{aligned}$$

It just so happened that  $\mathbb{E}[Y|X = x]$  above does not depend on what  $x$  is. In general,  $\mathbb{E}[Y|X = x]$  is parameterized by  $x$  (but we cannot have  $y$  appearing in the expression).

**Example:** consider again the joint pdf  $f(x, y) = x + y$  with the support given by  $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ . Recall that the conditional density is  $f_{Y|X=x}f(y|x) = \frac{2(x+y)}{1+2x}$  for  $(x, y) \in [0, 1]^2$ .

$$\begin{aligned} \mathbb{E}[Y|X = x] &= \int_0^1 y f_{Y|X=x}f(y|x) dy \\ &= \int_0^1 y \frac{2(x+y)}{1+2x} dy \\ &= \int_0^1 \frac{2xy}{1+2x} + \frac{2y^2}{1+2x} dy \\ &= \frac{x}{1+2x} + \frac{2}{3(1+2x)} = \frac{2+3x}{3+6x} \end{aligned}$$

We can see that  $\mathbb{E}[Y|X = x]$  is a function of  $x$ . Now let  $\mathbb{E}[Y|X = x] = f(x)$ . Then we define  $\mathbb{E}[X|Y]$  to be the random variable obtained by the transformation  $f(X)$ . As such,  $\mathbb{E}[Y|X]$  is a scalar random variable that has the same probability space as  $X$ .

$E[X|Y = y]$  is treated as a constant. We check that  $\mathbb{E}[Y|X = 0] = 2/3$ ,  $\mathbb{E}[Y|X = 1] = 5/9$ .  $\mathbb{E}[Y|X = x]$  is decreasing in  $x$ , what is the geometric intuition behind this? Now consider  $\mathbb{E}[Y|X]$ , which is a random variable defined by the transformation  $Z = \frac{2+3X}{3+6X}$ . We can then derive the pdf of  $\mathbb{E}[Y|X]$  from the marginal pdf  $f_X(x)$ .

Conditional expectation is very important and useful later on. Suppose  $Y$  is an outcome variable of interest, and  $X$  is a variable that can be used to predict  $Y$ . An excellent predictor of  $Y$  as a function of  $X = x$  is  $\mathbb{E}[Y|X = x]$ . This is optimal in a formal way.<sup>1</sup> For instance,  $Y$  is the transaction price of a house in the neighborhood and  $X$  is the square footage of the house. Then we can predict the price of a house when the square footage is 1000 as  $\mathbb{E}[Y|X = 1000]$ .

## 5. Independence

If  $X \sim f_X(x)$  and  $Y \sim f_Y(y)$  are independent, then the joint pdf of  $(X, Y)$  is:

<sup>1</sup>If your loss function is a mean-squared error. That is, let  $f(X) = \mathbb{E}[Y|X]$ , then  $f(X)$  minimizes the mean-squared error  $\mathbb{E}[(Y - h(X))^2]$  among all possible functions  $h(X)$ .

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Furthermore, if a joint pdf  $f_{X,Y}(x, y)$  can be factored as:

$$f_{X,Y}(x, y) = g(x)h(y)$$

Then  $X$  and  $Y$  are independent random variables.

**Example:** consider again the joint pdf  $f(x, y) = 6xy^2$  with the support on the unit square. Are  $X$  and  $Y$  independent? What about  $f(x, y) = 1$  with the support on the unit square?

Consider the pdf  $f(x, y) = 2$  with support on the triangle  $\{(x, y) \in [0, 1]^2 : x + y \leq 1\}$ . Are  $X$  and  $Y$  independent?

## 6. Covariance and correlation

The covariance between  $X$  and  $Y$  is:

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

Remember  $\mathbb{E}[XY] = \int \int xyf(x, y) dx dy$ .

The correlation between  $X$  and  $Y$  is:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Which is bounded between  $[-1, 1]$ .

A useful result is:

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$$

Show that when  $X$  and  $Y$  are independent, then  $\text{Cov}(X, Y) = 0$ . However the converse is not necessarily true! Zero covariance does not imply independence. Covariance only measures (roughly) a linear relationship between  $X$  and  $Y$ . We can easily construct a random variable  $X$  such that  $\text{Cov}(X, X^2) = 0$  (for example, any random variable such that the first and the third moments are zero), which means that covariance cannot capture non-linear relationship between random variables.

Instead, it is a good idea to always plot the scatterplot and inspect any non-linearity in the scatterplots.

**Example:**

Consider the joint pdf  $f(x, y) = 6xy^2$  with the support on the unit square. Recall that  $\mathbb{E}[XY] = \int_0^1 \int_0^1 xy 6xy^2 dx dy = \frac{1}{2}$ . Moreover,  $\mathbb{E}[X] = \frac{2}{3}$  and  $\mathbb{E}[Y] = \frac{3}{4}$ . Therefore,  $\text{Cov}(X, Y) = 0$ .

Similar calculations can be done for the discrete case:

$$\mathbb{E}[XY] = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} xy P(X = x, Y = y)$$

Finally, recall the joint pdf  $f(x, y) = x + y$  with the support on  $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ . We expect the covariance between  $X$  and  $Y$  to be positive.

Previously, we found that the marginal density of  $Y$  is  $f_Y(y) = \frac{1}{2} + y$  for  $y \in [0, 1]$ . As such,  $\mathbb{E}[Y] = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$ .

$$\begin{aligned} \mathbb{E}[XY] &= \int_0^1 \int_0^1 xy f(x, y) dx dy \\ &= \int_0^1 \int_0^1 xy(x + y) dx dy \\ &= \frac{1}{3} \end{aligned}$$

Therefore the covariance between  $X$  and  $Y$  is  $-\frac{1}{144}$ , which is negative. This number seems small, because it has not been normalized with the scale of  $(X, Y)$ . We can also show that  $\text{Var}(Y) = 11/144$ , and  $\text{Var}(X) = 11/144$ .<sup>2</sup> Hence, the correlation between  $(X, Y)$  is  $-\frac{1}{11}$ .

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<sup>2</sup>Note that the pdf is symmetric in  $x$  and  $y$ .